CHUKA UNIVERSITY

UNIVERSITY EXAMINATIONS

EXAMINATION FOR THE AWARD OF DEGREE OF MASTERS IN PURE MATHEMATICS

MATH 814: OPERATOR THEORY

STREAMS: PART TIME **TIME: 3 HOURS**

DAY/DATE: WEDNESDAY 7/4/2021 **2.30 PM – 5.30PM**

INSTRUCTIONS:

- Answer ANY three questions
- Sketch maps and diagrams may be used whenever they help to illustrate your answer
- Do not write on the question paper
- This is a **closed book exam**, No reference materials are allowed in the examination room
- There will be **No** use of mobile phones or any other unauthorized materials
- Write your answers legibly and use your time wisely

QUESTION ONE (20 MARKS)

a) State and prove the Existence Theorem for non-linear operator. (5 marks)

b) Let $T: l_2 \to l_2$ be a bounded linear operator on l_2 defined by $T(x_1, x_2, x_3, \ldots) = (-x_1, -x_2, -x_3, \ldots)$ for every $x \in \{x_i\}_{i=1}^{\infty} \in l_2$. Find the Eigen values of T. (3 marks)

c) Let P_1 and P_2 be projections on a Hilbert space H onto Y_1 and Y_2 respectively. If $P_1 P_2 = P_2 P_1$, show that $P = P_1 + P_2 - P_1 P_2$ is a projection. (4 marks)

d) Prove that the set of distinct nonzero Eigen values $\{\lambda_n\}$ of a self adjoint compact operator T is either finite or $\lim_{n \to \infty} \lambda_n = 0$. (4 marks)

e) Given the matrix $X = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ find the spectrum of $X, \delta(X)$ and the resolvent set of $X, \rho(X)$. (4 marks)

QUESTION TWO (20 MARKS)

a) Prove that if P is a projection then there exists a closed linear subspace $M\, of\, H$ such that $P = P_M$ or $P_M(H) = M$. (5 marks)

b) If a sum $P_1 + P_2 + P_3 + \ldots + P_n$ of projections $P_j : H \to H$ is a projection, show that $¿|P_1| \vee ¿^2 + ¿|P_2| \vee ¿^2 + ¿|P_3| \vee ¿^2 + \ldots + ¿|P_n| \vee ¿^2 ¿¿¿¿.$ (4 marks)

c) Let H be a Hilbert space and M be a linear closed subspace of H ad $y \in H¿$ then show that $¿ y - py, x > ¿ 0$ for all $x \in M$. (4 marks)

d) Prove S is a bounded linear operator on a Banach Space X and

$\|S\| < \|\lambda\|, S_\lambda = (\lambda I - S)^{-I}$ is bounded then $S_\lambda = \sum_{n=0}^{\infty} \frac{S^n}{\lambda^{n+1}} \wedge \|S_\lambda\| \leq \frac{1}{|\lambda| - ¿|S| \vee ¿¿}$. (5 marks)

e) State without proof Lax- Milgram Lemma. (2 marks)

QUESTION THREE (20 MARKS)

a) Prove that a bounded linear operator $P : H \to H$ on a Hilbert space is a projection if and only if P is self adjoint and idempotent. (8 marks)

b) Prove that if the resolvent set, $\rho(T)$ of a bounded linear operator T on a complex Banach space X is open then the spectrum $\sigma(T)$ is closed. (5 marks)

c) Let $T, S \in B(X)$. Show that if T is invertible and $¿ \vee T - S \vee ¿ < \frac{1}{¿ \vee T^{-I} \vee ¿}$ then S is invertible.

(3 marks)

d) Let $T \in B(X)$. Show that $T T^{¿}$ and $T^{¿} T$ are self adjoint and positive and the spectra of $T T^{¿}$ and $T^{¿} T$ are real and does not contain negative values. (4 marks)

QUESTION FOUR (20 MARKS)

a) Let H be a Hilbert space and let $T \in B(H)$

i. Define the numerical range of T, denoted by $Num(T)$. (2 marks)

ii. Prove that $¿<Tx,x>¿ \geq C¿|x|\vee¿^2¿$ for all $x \in H$ and $C>0$ then T is invertible and

$¿|T^{-1}|\vee \leq \frac{1}{C}$. (3 marks)

iii. Prove that $\delta(T) \subseteq \overline{Num(T)}$. (5 marks)

b) Find the Neumann Series solution of the integral equation $f(x)=x+\frac{1}{2}\int_{-1}^{1}(t-x)(t)dt$

(6 marks)

c) Show that the bounded linear operator T is normal *iff* $T=A+iB$ where $A \wedge B$are self adjoint operators with $AB=BA \wedge T\,T^{¿}=A^2+B^2$. (4 marks)

………