CHUKA UNIVERSITY

UNIVERSITY EXAMINATIONS

FOURTH YEAR EXAMINATION FOR THE DEGREE OF BACHELORS OF EDUCATION SCIENCE/ARTS AND BACHELOR OF SCIENCE MATHEMATICS

MATH 403: MEASURE THEORY

STREAMS: B. ED (SCIENCE/ARTS), BSC. MATH **TIME: 2 HOURS**

DAY/DATE: TUESDAY 21/09/2021 **8.30 A.M. – 10.30 A.M.**

INSTRUCTIONS:

- Answer question **ONE** and **TWO** other questions
- This is a **closed book exam**, No reference materials are allowed in the examination room
- There will be **No** use of mobile phones or any other unauthorized materials

QUESTION ONE: (30 MARKS)

a) Prove that if $\mu^*(A)=0$ then for any set B, $\mu^*(A\cup B)=\mu^*(B)$. (3 marks)

b) Prove the following properties of an outer measure μ^*

   i. $\mu^*(\phi)=0$ (2 marks)

   ii. $\mu^*(\{x\})=0$ (2 marks)

   iii. If $A\subseteq B$ then $\mu^*(A)\leq\mu^*(B)$ (2 marks)

c) Using the properties of outer measure, prove that

   i. The unit interval $I=[0,1]$ is not countable

   ii. The outer measure of all the irrational numbers in $I=[0,1]$ is 1. (4 marks)

d) Define a measurable function and show that the characteristic function on a measurable set is measurable. (3marks)

e) Differentiate a finite measure and sigma finite measure. (2marks)

f) Show that the space (R,B,μ) is not complete, where μ is the restriction of Lebesgue measure to the Borel sets. (4 marks)

g) Shoe that if $A \cup B$ is measurable whenever A and B are measurable, then $A \cap B$ is measurable (4marks)

h) Show that the integral is monotone i.e.

i).If $f, g \in M^{+}(X, x)$ and $f \leq g$ then $\int f d\mu \leq \int g d\mu$ (2 marks)

ii) If $f \in M^{+}(X, x)$ and $E, F \in x$ such that $E \subset F$ then $\int_{E} f d\mu \leq \int_{F} f d\mu$ (2marks)

QUESTION TWO: (20 MARKS)

a) Define a Lebesgue measurable subset of R. (2 marks)

b) Define a Lebesgue non-measurable. Hence show that if a set F is Lebesgue non-measurable, there exist a proper subset A of F such that $0 < \mu^{*}(A) < \infty$ (4 marks)

c) Show that if $\mu^{*}(A) = 0$, then A is measurable hence or otherwise show that a countable set is measurable. (5 marks)

d) Prove that measurable sets form a sigma algebra (9 marks)

QUESTION THREE: (20 MARKS)

a) Let X, Y be non-void sets and $f: X \to Y$ be a function. Let $\beth$ be the σ- algebra of subsets of Y and let $\mathfrak{x} = \{f^{-1}(E): E \in \beth\}$. Prove that then $\mathfrak{x}$ is the σ- algebra of subsets of X (6marks)

b) Let A be an uncountable subset of R and define a class Ω of subsets of A as follows:

$\Omega = \{E \subseteq A$ if E is countable or A-E is countable$\}$

i. Show that Ω is a sigma algebra (6 marks)

ii. Define a function $f: \Omega \to R$ as $f(E) = \begin{cases} 0 if E = countable \\ 1 otherwise \end{cases}$.

Show that f is a measure (8 marks)

QUESTION FOUR: (20 MARKS)

a) Let f be a measurable function, prove that the following conditions are equivalent

i. $\{x: f(x) > \alpha\}$ is Lebesgue measurable $\forall \alpha \in R$

ii. $\{x: f(x) \geq \alpha\}$ is Lebesgue measurable $\forall \alpha \in R$

iii. $\{x: f(x) < \alpha\}$ is Lebesgue measurable $\forall \alpha \in R$

iv. $\{x: f(x) \leq \alpha\}$ is Lebesgue measurable $\forall \alpha \in R$ (8 marks)

b) Show that if f is measurable, then so are the functions f^{2} and $|f|$.

Is the converse true? Verify (4 marks)

c) (i)State without prove the monotone convergence theorem (M.C.T) (2 marks)

(ii)Show that the sequence $f_n(x) = \frac{1}{n}\chi_{[0,n]}$ for $n \in N$ uniformly converges to $f = 0$ (2 marks)

(iii)Show that M.C.T does not apply in the sequence $f_n(x) = \frac{1}{n}\chi_{[0,n]}$ for $n \in N$. Explain your answer. (4 marks)

QUESTION FIVE: (20 MARKS)

a) (i)Show that intervals of the form $(a,b): a < b$ and $a,b \in R$ are Lebesgue measurable (8 marks)

(ii) Hence conclude that the sets $[a,b],[a,b),(a,b]$ are Lebesgue measurable (6 marks)

b) Let $\{E_n\}$ be a sequence of measurable sets with the properties $E_n \supseteq E_{n+1}$ and $\mu(E_1) < \infty$.

Prove that $\mu(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} E_n$ (6 marks)

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