CHUKA UNIVERSITY

UNIVERSITY EXAMINATIONS

THIRD YEAR EXAMINATION FOR THE AWARD OF DEGREE OF BACHELOR OF EDUCATION SCIENCE/ARTS, BACHELORS OF SCIENCE, BACHELORS OF ARTS (MATHS-ECON)

MATH 301: LINEAR ALGEBRA II

STREAMS: `` AS ABOVE`` **TIME: 2 HOURS**

DAY/DATE: MONDAY 29/03/2021 **11.30 A.M. – 1.30 P.M.**

INSTRUCTIONS:

- Answer Question **ONE** and any other **TWO** Questions
- Sketch maps and diagrams may be used whenever they help to illustrate your answer
- Do not write anything on the question paper
- This is a **closed book exam**, No reference materials are allowed in the examination room
- There will be **No** use of mobile phones or any other unauthorized materials
- Write your answers legibly and use your time wisely.

QUESTION ONE: (30 MARKS)

a) Given that $A=\begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$, find the eigenvalues of A^6 (4 marks)

b) Find the symmetric matrix that correspond to the following quadratic form
$q(x,y,z)=xy+y^2-4xz+z^2$ (2 marks)

c) Show that if $A=\begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix}$, then A is a zero of the function $f(t)=t^2-5t+7$ (3 marks)

d) Let A be an nxn matrix over a field K. show that the mapping $f(X,Y)=X^{T}AY$ is a bilinear form on K^{n} (2 marks)

e) Let A be a diagonalizable matrix and let P be an invertible matrix such that $B = P^{-1}AP$, that is B is the diagonal form of A .Prove that $A^k = PB^kP^{-1}$ where k is a positive integer. (3 marks)

f) State how elementary row operations affect the determinant of a square matrix. Hence or otherwise show that if one row of a square matrix is a scalar multiple of another row, the determinant of that matrix is zero. (5 marks)

g) Determine whether the quadratic form $q(x,y,z)=x^2-4xz+2y^2-4yz+7z^2$ is positive definite (4 marks

h) Find the minimal polynomial of the matrix $A=\begin{bmatrix}2&1&0&0\\0&2&0&0\\0&0&1&1\\0&0&-2&4\end{bmatrix}$ (4 marks)

QUESTION TWO (20 MARKS)

a) Let A be a square matrix of order n>2. If λ_1 and λ_2 are two distinct eigenvalues of A with corresponding eigenvectors v_1 and v_2 respectively then show that v_1 is orthogonal to v_2.
(4 marks)

b) Let A be the matrix

$$\begin{bmatrix}2&2&-2\\2&-1&4\\-2&4&-1\end{bmatrix}$$

i. Find an orthogonal matrix P such that $D=P^{T}AP$ is diagonal. Find D. (8 marks)

ii. Apply diagonalization algorithm of congruent matrices to obtain a matrix P such that $D'=P'^{T}\ AP'$. What is D in this case? (6 marks)

iii. Differentiate P with P' and D and D' from i. and ii. Above (2 marks)

QUESTION THREE (20 MARKS)

a) Let f be a bilinear form on R^2 defined by
$f[(x_1,x_2),(y_1,y_2)]=3x_1y_1-2x_1y_2+4x_2y_1-x_2y_2$. Find

i. The matrix A of f in the basis $\{u_1=(1,-1),u_2=(1,1)\}$

ii. The matrix B of f in the basis $\{v_1=(1,2),v_2=(2,-1)\}$

iii. The change of basis matrix P from the basis $\{u_i\}$ to the basis $\{v_i\}$ and verify that $B=P^{T}AP$. (12 marks)

i. Consider the quadratic form $q(x,y)=3x^2+2xy-y^2$ and the linear substitution $x=s-t$ and $y=s+t$

i. Rewrite $q(x, y)$ in matrix notation and find the matrix notation and find the matrix A representing $q(x, y)$ (1 mark)

ii. Rewrite the linear substitution using matrix notation and find the matrix P corresponding to the substitution (3 marks)

iii. Write the quadratic form $q(s, t)$ (2 marks)

iv. Verify part iii above using direct substitution (2 marks)

QUESTION FOUR (20 MARKS)

a) Given that $A = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 5 & -3 & 1 \\ 1 & 6 & -2 & 4 \\ 0 & 1 & -3 & 7 \end{bmatrix}$ determine the number n_k and the sum S_k of principal minors of order 1, 2 and 4. (6 marks)

b) Let $A = \begin{bmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{bmatrix}$

i. Find the characteristic polynomial of A. (3 marks)

ii. Find all the eigenvalues of A and their corresponding eigenvectors. (4 marks)

iii. Determine the matrix P if it exists such that $D = P^{-1}AP$ where D is diagonal. (1 mark)

iv. Comment about the minimal polynomial of A (2 marks)

c) State Cayley-Hamilton theorem for a linear operator and verify the theorem using a linear operator $T : R^2 \to R^2$ defined by $T(x_1, x_2) = (4x_1 - 3x_2, x_1 + 5x_2)$ (4 marks)

QUESTION FIVE (20 MARKS)

a) Given a 2x2 matrix of the form $A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$, show that A is diagonalizable. (5 marks)

b) Let V ⊂ R[x] be the polynomials of degree 2 or less. let T: V → V be the linear operator T(p(x)) = (x + 1)p ′ (x). Find the minimal polynomial of T and conclude whether or not T is diagonalizable (6 marks)

c) Apply Gram-Schmidt orthogonalization process to the basis $B = \{1, x, x^2\}$ in $P_2(x)$ with the inner product $\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)dx$ to obtain an orthogonal set (5 marks)

d) Find an orthonormal basis for the solution space to the orthonormal system of equations below

$$x_1 + x_2 + 7x_4 = 0$$
$$2x_1 + x_2 + 2x_3 + 6x_4 = 0$$
(4 marks)
