

CHUKA



UNIVERSITY

## UNIVERSITY EXAMINATIONS

**SECOND YEAR EXAMINATION FOR THE DEGREE OF BACHELORS OF  
SCIENCE MATHEMATICS AND BACHELOR OF EDUCATION (SCIENCE)**

MATH 405: ALGEBRA II

STREAMS: AS ABOVE

TIME: 2 HOURS

DAY/DATE: MONDAY 27/09/2021

11.30 A.M. – 1.30 P.M.

**INSTRUCTIONS:**

- Answer Question **ONE** and any other **TWO** Questions
- Sketch maps and diagrams may be used whenever they help to illustrate your answer
- Do not write anything on the question paper
- This is a **closed book exam**, No reference materials are allowed in the examination room
- There will be **No** use of mobile phones or any other unauthorized materials
- Write your answers legibly and use your time wisely

**QUESTION ONE (30 MARKS)**a) Find two ideals  $I$  and  $J$  in the ring  $Z$  of integers such that

- i.  $I \cup J$  is an ideal
- ii.  $I \cup J$  is NOT an ideal (4 marks)

b) The addition and part of the multiplication table for the ring  $R = \{a, b, c\}$  are given below. Use the distributive laws to complete the multiplication table below

+	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

*	a	b	c
a	a	a	a
b	a	c	
C	a		

(4 marks)

b) For any element  $a \in \mathbb{Z}$ ; the ring of integers let  $[a]_6$  denote  $[a] \in \mathbb{Z}_6$  and  $[a]_2$  denote  $[a] \in \mathbb{Z}_2$

i. Prove that the mapping  $\phi : \mathbb{Z}_6 \rightarrow \mathbb{Z}_2$  defined by  $\phi([a]_6) = [a]_2$  is a homomorphism

ii. Find  $\ker \phi$  (6 marks)

c) Working in  $\mathbb{Q}[x]$ , find the highest common factor of  $x^3 + x^2 - 8x - 12$  and  $x^3 + 5x^2 + 8x + 4$  and express it as a linear combination of the two functions (5 marks)

d) If  $R$  is a commutative ring with identity, show that  $R[x]$  is also a commutative ring with identity

(5 marks)

e)

i. Show that the set of  $R$  consisting of all matrices of the form

$$T = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in \mathbb{Z} \right\} \text{ is a non-commutative ring with unity.}$$

ii. Which elements of  $T$  are invertible?

iii. Find if  $I = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}$  is an ideal of  $T$  (6 marks)

**QUESTION TWO (20 MARKS)**

a) Consider the set  $R = \{[0], [2], [4], [6], [8], [10], [12], [14], [16]\} \subseteq \mathbb{Z}_{18}$ .

i. Construct addition and multiplication tables for  $R$  using operations as defined in  $\mathbb{Z}_{18}$  (2 marks)

ii. Show that  $R$  is a commutative ring with unity. (2 marks)

iii. Show that  $R$  a subring of  $\mathbb{Z}_{18}$  (2 marks)

- iv. Does  $R$  have zero divisors? (1 mark)
- v. Is  $R$  a field? If yes illustrate each element with its inverse (1 mark)
- b) Let  $R$  be a ring in which the only ideals are  $\{0\}$  and  $R$ . Prove that  $R$  is a field (6 marks)
- c) Let  $I$  be an ideal of a commutative ring  $R$  with unity. Show that if  $I$  contains a unit element, then  $I=R$  (6 marks)

**QUESTION THREE (20 MARKS)**

- a) Let  $F$  be a field, and let  $f(x)$  and  $g(x)$  be polynomials in  $F[x]$  where  $F$  is a field
  - i. Prove that  $\deg(fg) = \deg(f) + \deg(g)$ . (4 marks)

Consider the polynomials  $f(x) = 2x^2 + 3x + 3$  and  $g(x) = 3x + 1$  in the polynomial ring  $Z_6[x]$

s. Find:

- i.  $\deg(f)$
- ii.  $\deg(g)$
- iii.  $\deg(fg)$
- iv. why is the theorem above not satisfied (4 marks)
- b) Let  $X$  be a non-empty set and  $R$  be the set of all subsets of  $X$ . Define addition and multiplication in  $R$  as follows

$$A + B = A \cup B - A \cap B$$

$$A * B = A \cap B$$

For all  $A \in R$  define a function  $f : R \rightarrow Z_2$  as  $f(x) = \begin{cases} \bar{1} & \text{if } x \in A \\ \bar{0} & \text{otherwise} \end{cases}$

- i. Show that  $A + \phi = A$  and  $A + A = \phi$  (5 marks)
- ii. Show that  $f$  is a homomorphism of rings (7 marks)

**QUESTION FOUR (20 MARKS)**

- a) Let  $U$  be a fixed non-empty set and  $R$  be the set of subsets of  $U$  with addition and multiplication defined by  $A + B = A \cup B$  and  $A \times B = A \cap B$ . Verify whether or not  $(R, +, \times)$  is a ring. (6 marks)
- b) Let  $F$  be a field, and  $f(x)$  a non-zero polynomial in  $F[x]$ . Prove the following
  - i. If  $g(x) \in F[x]$  is an associate of  $f(x)$ , then  $\deg(g) = \deg(f)$ . (4 marks)

- ii. There exists a unique monic polynomial that is an associate of  $f(x)$ . ( 4 marks)
- c) Let  $I$  and  $J$  be ideals in the ring  $Z$  of integers, Verify whether or not
  - i.  $I + J = \{x + y : x \in I, y \in J\}$  is an ideal
  - ii.  $I \cap J$  is an ideal ( 6 marks)

**QUESTION FIVE (20 MARKS)**

- a) i. Use the Euclidean Algorithm to find  $hcf(x^3 + 2x^2 - x - 2, x^2 - 4x + 3)$  in  $Q[x]$ . (4 marks)
- ii. Hence, or otherwise, find polynomials  $s, t$  in  $Q[x]$  for which
 
$$x - 1 = s(x^3 + 2x^2 - x - 2) + t(x^2 - 4x + 3)$$
 (4 marks)
- iii. find  $lcm(x^3 + 2x^2 - x - 2, x^2 - 4x + 3)$  (4 marks)
  
- b) prove that in a ring of integers, every ideal is a principal ideal (8 marks)

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