CHUKA **UNIVERSITY**

UNIVERSITY EXAMINATIONS

FOURTH YEAR EXAMINATION FOR THE AWARD OF DEGREE OF BACHELOR OF EDUCATION SCIENCE / ARTS, BSC. MATHEMATICS

MATH 409: FUNCTIONAL ANALYSIS

STREAMS: AB5 / AB1 ; EB6 Y4S2

TIME: 2 HOURS

DAY/DATE : WEDNESDAY 22 /09/ 2021 **2.30 PM – 4.30 PM**

INSTRUCTIONS TO CANDIDATES:

- Answer question **ONE** and **TWO** other questions
- Do not write on the question paper
- This is a **closed book exam**, No reference materials are allowed in the examination room
- There will be **No** use of mobile phones or any other unauthorized materials
- Write your answers legibly and use your time wisely

QUESTION ONE: (30 MARKS)

(a) Distinguish the following terms

i. A convergence sequence and a Cauchy sequence (2marks)

ii. Holders Inequality and Minikowsk`s Inequality (2marks)

iii. A Hamel base and Schauder Basis (2marks)

iv. A semi-norm and a para-norm (4marks)

v. An Iteration and a contraction mapping (2marks)

vi. A complete space and a compact space (2 marks)

(b) (i) Define a Banach space. Hence give any two examples of Banach spaces (2 marks)

(ii) When are two norms said to be equivalent on a vector space? (2marks)

(c) (i) When are two normed linear spaces said to be Isometrically Isomorphic? (1mark)

(ii) Hence show that the spaces $C_O{}^* \sim \ell_1$ are Isometrically Isomorphic (3 marks)

(d) Prove that for a weakly convergent sequence, its limit point of is unique (3 marks)

(e) (i) Define a sesquillinear functional on normed linear spaces $X \ and \ Y$ (2 marks)

(ii) Hence state without proof the Riesz's Representation Theorem (3 marks)

QUESTION TWO: (20 MARKS)

(a) State the Parallelogram law as used in inner product spaces. Hence using an appropriate example, illustrate that all Banach spaces are not necessarily inner product spaces. (6 marks)

(b) Define a fixed point of a mapping T of a set X. Give two cases that illustrate a fixed point mapping. (3marks)

(c) State and prove the Banach Fixed Point Theorem on a metric space$X$ (11marks)

QUESTION THREE: (20 MARKS)

(a) Prove that on the space of all sequences S, the mapping defined by

(i) $P_1(x) = \sup|x_n| \ \forall \, n \geq 1$ is a semi-norm (3marks)

(ii) $p(x) = |x| = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|x_k|}{1+|x_k|}$ is a paranorm (5marks)

(b) Define a norm on a linear space X. Hence show that the mapping $\|.\|: R^n \to R$ defined by

$\| x \|_\infty = \left(\left(\sum_{k=1}^{n} \| x_k \|^2 \right) \right)^{\frac{1}{2}}$ is a normed space. (12marks)

QUESTION FOUR: (20 MARKS)

(a) Let $T: X \to Y$ be a linear operator from a normed linear space X into a normed linear space Y, prove that T is continuous if and only if T is bounded (10 marks)

(b) Prove that strong convergence implies weak convergence, and with an appropriate counter example show that the converse is not necessarily true (10 marks)

QUESTION FIVE: (20 MARKS)

(a) Let $T: X \to Y$ be a linear operator from a normed linear space X into a normed linear space Y. Prove that :

(i) T is continuous iff T is bounded. (5 marks)

(ii) T is continuous at the origin implies that T is uniformly continuous on X (3 marks)

(b) Show that if $(T_n)_1^\infty$ be a sequence of bounded linear operators each defined on a Banach space X into a normed linear space Y such that for each $x \in X, \lim_{n\to\infty} T_n(x) = T(x)$ exists in Y, then T is a bounded linear operator from X into Y. (8marks)

(c)) Define a meager subset of a metric space X. Hence state without proof, Baire's Category Theorem (4marks)

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