CHUKA UNIVERSITY

UNIVERSITY EXAMINATIONS

FOURTH YEAR EXAMINATION FOR THE AWARD OF DEGREE OF BACHELOR OF SCIENCE IN MATHEMATICS

MATH 448: STOCHASTIC PROCESSES

STREAMS: BSC MATH **TIME: 2 HOURS**

DAY/DATE: FRIDAY 24/09/2021 **11.30 A.M – 1.30 P.M**

INSTRUCTIONS

Answer question ONE and any other TWO questions

Show your working clearly

Question One (30 MARKS)

a) Considering a birth – death processes, explain two methods of getting the mean and the variance of the process. (4 marks)

b) Consider a Fibonacci process given by $a_n = a_{n-1} + a_{n-2}$ where $a_0 = a_1 = 1$. Obtain the generating function for the process. (3 marks)

c) Classify the states of the following Markov chain

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$ (6 marks)

d) Define a stochastic process and give an appropriate example. (3 marks)

e) Consider the process $yt = A \cos wt + B \sin wt$ where A and B are independent random variables and W is a constant. Given $E(A) = E(B) = 0$ that $var(A) = var(B) = \sigma^2$. Show that the process is covariance stationary [NB: $Cov\ (A, B) = 0$] (5 marks)

f) Define the following terms

i. Absorbing state.

ii. Markov chain

iii. Ergodicity. (3 marks)

g) Let X have the distribution form (modified geometric or decapitated geometric) given by $pr\ (X = k) = q^{k-1}p$ where $k = 1,2,3 \ldots$ and $p + q = 1$. Show that the probability generating function X is given by $p(s) = \frac{ps}{1-qs}$ and that $E(x) = \frac{1}{p}$ and $Var(x) = \frac{q}{p^2}$ (6 marks)

Question Two (20 MARKS)

a) Let X have a pdf $\Pr(X = k) = P_k$ for $K = 0,1,2, \ldots$ with pgf $P(S) = \sum_k P_k S^k$ and $q_k = q_r = p_r(X > K) = p_{k+1} + p_{k+2} + \cdots$ Show that the generating function of $Q(S) = \frac{1-p(s)}{1-s}$

(7 marks)

b) Consider the difference-differential equation for a poisson process i.e;

$$p_n'(t) = -\lambda p_n(t) \pm \lambda p_{n-1}(t) : n \geq 1 \text{ and}$$

$$p_0'(t) = -\lambda p_0(t) : n = 0$$

Obtain the mean and the variance of the process. Take the initial conditions to be

$$p_n(0) = \begin{cases} 1; & n = 0 \\ 0: & otherwise \end{cases}$$ (13 marks)

Question Three (20 MARKS)

a) Let Y be the family size or the number of offspring's distribution.
Z_n be the size of the population at time n or the size of generation n, where n = 0, 1, 2, …
Let the $E(Y) = \mu$ and $Var(Y) = \sigma^2$, the mean and variance of the number of offspring of a single individual. Suppose that $\{Z_0, Z_1, Z_2, \ldots\}$ is a branching process with $Z_0 = 1$ (starting with a single individual).

Then proof that:

1) $E(Z_n) = \mu^n$
2) $Var(Z_n) = \begin{cases} n\sigma^2, & if\ \mu = 1 \\ \sigma^2 \mu^{n-1} \left(\frac{1-\mu^n}{1-\mu}\right), & if\ \mu \neq 1 \end{cases}$
3) Using (1) and (2) above and given that the family size $Y \sim$Binomial (25, 0.4) find the:
   i) Expected population size by the 25th generation, $[E(Z_{25})]$
   ii) Variance of the population by the 25th generation, $[Var(Z_{25})]$ (12 marks)

b) Consider the three states markov chain having a transition probability matrix

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Obtain the stationary distribution (8 marks)

Question Four (20 MARKS)

a) Classify the states of the following markov chain

$$\begin{bmatrix} 1/2 & 1/2 & 0 & 0 \ldots\ldots\ldots & 0 \\ 1/2 & 0 & 1/2 & 0 \ldots.\ldots & 0 \\ 1/2 & 0 & 0 & 1/2 \ldots\ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1/2 & 0 & 0 & 0 \ldots\ldots & 1/2 \end{bmatrix}$$

(13 marks)

b) Consider a birth death process with $z(t)$ being the population size at time t and $p_n(t)$ being the probability that the population is of size n at time t. Making the necessary assumptions, build up a model for $p_n\ (t + \Delta t)$ to give the final difference differential equations for the model. (7 marks)

Question Five (20 MARKS)

a) A credit union classifies automobile loans into one of four categories: the loan has been paid in full (F), the account is in good standing (G) with all payments up to date, the account is in arrears (A) with one or more missing payments, or the account has been classified as bad debt (B) and sold to a collection agency. Past records indicate that each month 10% of the accounts in good standing pay the loan in full, 80% remain in good standing and 10% become in arrears. Furthermore, 10% of the accounts in arrears are paid in full, 40% become accounts in good standing, 40% remain in arrears, and 10% are classified as bad debts.

   i) In the long run, what percentage of the accounts in arrears will pay their loan in full?
   ii) In the long run, what percentage of the accounts in good standing will become bad debts?
   iii) What is the average number of months an account in arrears will remain in this system before it is either paid in full or classified as a bad debt?
   (14 marks)

c) Let X and Y be independent Poisson random variables with parameters $\lambda \; and \; \mu$ respectively. Given that $Z = X + Y$; Find the:

   i) p.g.f of X and p.g.f of Y
   ii) p.g.f of Z (6 marks)
