CHUKA UNIVERSITY

UNIVERSITY EXAMINATIONS

FOURTH YEAR EXAMINATION FOR THE AWARD OF DEGREE OF BACHELOR OF

MATH 449: PROBABILITY THEORY

STREAMS: BSc. MATHEMATICS **TIME: 2 HOURS**

DAY/DATE: MONDAY 29/03/2021 **11.30 A.M. – 1.30 P.M.**

INSTRUCTIONS:

- Answer question **ONE** and **TWO** other questions
- Sketch maps and diagrams may be used whenever they help to illustrate your answer
- Do not write on the question paper
- This is a **closed book exam**, No reference materials are allowed in the examination room
- There will be **No** use of mobile phones or any other unauthorized materials
- Write your answers legibly and use your time wisely

QUESTION 1[30 MARKS]

(a) Define the following:
   i) Convergence in probability [2 Marks]
   ii) Convergence in distribution [2 Marks]

(b) Briefly describe Bernoulli's weak law of large numbers. [3 Marks]

(c) State and proof the Borel - Cantelli Lemma. [5 Marks]

(d) Differentiate between the σ −field and the Borel field [5 Marks]

(e) Consider a field denoted by $\mathcal{A}$ and let $A_1, A_2, \ldots, A_3 \in \mathcal{A}$. Show that $\bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} A_i \in \mathcal{A}$ [5 Marks]

(f) Define the indicator function $I_A(x)$ of a set A. [3 Marks]

(g) Given the sets A={2,3,4,5,6} and B={1,2,3,4,5,6,7}
obtain, $I_A(1), I_{B^c}(1), I_{AB}^{(1)}, I_{A-B}(3)\ I_{A\cup B}(7)$ [5 Marks]

QUESTION 2 [20 MARKS]

a) Consider a probability measure defined as $(\Omega, \mathcal{F}, \mathbb{P})$.
  i) Explain each of the elements defined in the above space [3 marks]

  ii) If $A \in \Omega$, explain all the properties of $\mathbb{P}$ using mathematical expressions. [4 marks]

b) Explain the term independence as used in probability theory. [3 marks]

c) Let X_n be i.i.d, $E(X_n) = \mu\ and\ Var(X_n) = \sigma^2$. Set $\bar{X} = \frac{1}{2}\sum_{i=1}^{n} X_i$, show that $\frac{1}{2}\sum_{i=1}^{n}(X_i - \bar{X})^2 \longrightarrow \sigma^2$ [10 marks]

QUESTION 3 [20 MARKS]

a) (Borel-Cantelli Lemma) Suppose $A_1, A_2, \ldots, A_n$ is a sequence of events
i) If $\sum \mathbb{P}(A_n) < \infty$ then, $\mathbb{P}(A_n, o.i) = \mathbb{P}(Lim\ Sup\ A_n) = 0$
ii) If $\sum \mathbb{P}(A_n) = \infty$ and $A_1, A_2, \ldots, A_n$ are independent then, $\mathbb{P}(A_n, o.i) = \mathbb{P}(Lim\ Sup\ A_n) = 1$.
Prove [10 marks]

b) Find E(X) when
  (i) X is the maximum of two dice rolls. [5 marks]

  (ii) X is the number of tosses of biased coin until head appears [5 marks]

QUESTION 4 [20 MARKS]

a) If two dice were rolled once and we are interested in the events where the two numbers that show up are equal (B_1), their sum are odd (B_2), their sums are 14 (B_3). Apply the concept of a probability to come up with $\Omega, \mathcal{F}\ and\ \mathbb{P}$ respectively for this experiment. [6 marks]

b) Suppose $X_1, X_2, \ldots$ is a sequence of random variables with $E(X_n) \longrightarrow \mu$ and $Var(X_n) \longrightarrow 0$. Show that $X_n \longrightarrow \mu$. [4 marks]

c) If A_n is increasing then $P(A_n)$ is increasing and $\lim P(A_n) = P(\cup_{n=1}^{\infty} A_n)$.
Prove [10 marks]

QUESTION 5 [20 MARKS]

Let $X_1, X_2, \ldots$ be i.i.d with finite $E(X_i) = \mu$ and finite variance $Var(X_i) = \sigma^2$. Let $S_n = X_1 + X_2 + \cdots + X_n$, then the sequence $\frac{S_n - n\mu}{\sigma}$ converges to the standard normal in the distribution.(Hint: Central Limit Theorem).Prove. [20 marks]
