Spectral Picture Of Almost Similar Operators

Eric Gitonga Muriithi
Department of Physical Sciences,
Chuka University
Chuka, Kenya
Gitongaeric420@gmail.com

Sammy W. Musundi
Department of Physical Sciences,
Chuka University
Chuka, Kenya
swmusundi@chuka.ac.ke

Bernard M. Nzimbi
School of Mathematics
University of Nairobi,
Nairobi, Kenya
nzimbi@uonbi.ac.ke

ABSTRACT-Various results that relate to almost similarity and other classes of operators such as isometry, normal, unitary and compact operators have been extensively discussed. In this paper, we describe the spectral picture of almost similar operators. To be more specific we will describe the spectrum, the spectral radius, the numerical radius as well as the norm of almost similar operators.

Keywords: **Almost similarity; self-adjoint; similar operators**

1. INTRODUCTION

Let H denote a Hilbert space and B(H) denote the Banach algebra of bounded linear operators. If $\in B(H)$, then T^* denote self adjoint of T, while $\sigma(T)$, $r(T)$, $\omega(T)$, $\| T \|$, $W(T)$, stands for the spectrum, spectral radius, numerical radius, norm and numerical range of T, respectively. Recall that an operator $T \in B(H)$ is Self-adjoint if $T^* = T$.
A projection if $T^2 = T$ and $T^* = T$
Two operators $T \in B(H)$ and $S \in B(K)$ are said to be similar if there exists an invertible operator $N \in B(H, K)$ such that $NT = SN$ or equivalently $T = N^{-1}SN$. Jibril (1996) introduced the concept of almost similar and it was studied further by (Nzimbi *et al*., 2008) and (Musundi *et al*., 2013)
Two operators A and B are said to be *almost similar* if there exists an invertible operator N such that the following conditions are satisfied:

$$A^*A = N^{-1}(B^*B)N$$
$$A^* + A = N^{-1}(B^* + B)N$$

2. MAIN RESULTS

In this section we explicitly establish the spectral picture of almost similar operators.

2.1 Spectral picture of almost similar operators.

To establish the spectral picture of almost similar operators we need the following results:

Lemma 2.1.1: Let T and S be almost similar self-adjoint operators, then T and S are similar
Proof. Let N be an invertible operator such that

$$T^*T = N^{-1}(S^*S)N \quad (1)$$
$$S^* + S = N^{-1}(S^* + S)N \quad (2)$$

Since

$$T^* = T$$

and

$$S^* = S \text{ i.e. self-adjoint}$$

then equality (1) becomes

$$T^2 = N^{-1}(S^2)N \quad (3)$$

and equality (2) becomes

$$2T = N^{-1}(2S)N \quad (4)$$

Dividing both sides of equality (4) by 2, equation (4) collapse to the equality $T = N^{-1}(S)N$, i.e. T is similar to S.
Next we state the Halmos Lemma which is essential in proving the subsequent results.

Lemma 2.1.2: (Halmos, 1967)
Suppose that A and B are similar operators on a Hilbert space H, then A and B have the same

i. Spectrum
ii. Point spectrum
iii. Approximate point spectrum

Theorem 2.1.3: If T and S are almost similar projections then

$$\sigma(T) = \sigma(S).$$

Proof. We first need to show that S and T are similar. Suppose that N is an invertible operator such that

$$T^*T = N^{-1}(S^*S)N \qquad (5)$$

and

$$T^* + T = N^{-1}(S^* + S)N \qquad (6)$$

Since S and T are projections, then they are self adjoint. According to Lemma 2.1.1 equality (5) and (6) collapse to the equality

$$T = N^{-1}SN$$

i.e. T is similar to S and by Lemma 2.1.2 above we conclude that

$$\sigma(T) = \sigma(S)$$

Since projection operators are self adjoint then theorem 2.1.3 simplifies to the result below.

Corollary 2.1.4: If T and S are almost similar self-adjoint operators then

$$\sigma(T) = \sigma(S).$$

Proof

Suppose T and S are self-adjoint

$$(T = T^*,\ S = S^*)$$

and

$$T^*T = N^{-1}(S^*S)N$$
$$T^* + T = N^{-1}(S^* + S)N$$

That is T and S are similar according to theorem 2.1.3 and therefore have equal spectrum. i.e.

$$\sigma(T) = \sigma(S)$$

Remark 2.1.5: Almost similarity does not preserve the spectrum of operators because almost similarity does not in general imply similarity.

Definition 2.1.6: Let $\lambda_1 \ldots\ldots \lambda_n$ be the (real and complex) eigenvalue of matrix $T \epsilon \mathbb{C}^{n\times n}$. Then its *spectral radius* denoted by *r*(T) is defined as

$$r(T) = \max\{|\lambda_1| \ldots .|\lambda_2|\}$$

Theorem 2.1.7: If T and S are self-adjoint operators which are almost similar then

$$r(T) = r(S)$$

Proof

Since T and S are almost similar self-adjoint operators, then by Corollary 2.1.4

$$\sigma(T) = \sigma(S)$$

and by definition 2.1.6, then

$$r(T) = r(S).$$

Having seen that $r(T) = r(S)$ in this part of our section 2.1, we now need to know whether two almost similar self-adjoint operators have equal norms and to know these we need the following.

Proposition 2.1.8: (Halmos, 1967)
Let $T \epsilon B(H)$ be a self-adjoint operator. Then

$$\omega(T) = r(T).$$

Proposition 2.1.9: (Skoufranis, 2014)
Let $T \epsilon B(H)$ be a self-adjoint operator. Then

$$\omega(T) = \| T \|.$$

Proposition 2.1.10: Let $T, S \epsilon B(H)$ be almost similar self-adjoint operator. Then
$\| T \| = \| S \|$.

Proof. Since T and S are almost similar self-adjoint operators, then by Corollary 2.1.4

$$\sigma(T) = \sigma(S)$$

and by definition 2.1.6, then

$$r(T) = r(S).$$

Using Proposition 2.1.8, then

$$\omega(T) = \omega(S).$$

and so by Proposition 2.1.9

$$\begin{aligned} \| T \| &= \omega(T) \\ &= \omega(S) \\ &= \| S \| \end{aligned}$$

i.e.

$$\| T \| = \| S \|$$

Remark 2.1.11: We now conclude that almost similar self-adjoint operators have equal norms.

Definition 2.1.12: An operator $T \in B(H)$ is said to be normaloid if

$$\omega(T) = \| T \|$$

Theorem 2.1.13: let $\boldsymbol{T}, \boldsymbol{S} \in \boldsymbol{B}(\boldsymbol{H})$ be self-adjoint and almost similar normaloid operator. Then

$$\boldsymbol{\omega}(\boldsymbol{T}) = \boldsymbol{\omega}(\boldsymbol{S}).$$

Proof

Since the operators are normaloid and self-adjoint. Then

$$\begin{aligned} \omega(T) &= \| T \| \\ &= \| S \| \\ &= \omega(S). \end{aligned}$$

Its evidence that two self-adjoint almost similar normaloid operators have equal numerical radius.

Example

$$T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Both operators are self-adjoint and a simple computation shows that A and B are almost similar with

$$N = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

A simple computation shows that

$$\sigma(T) = \sigma(S)$$
$$= \{-1,1\}$$

This implies that

$$r(T) = r(S)$$
$$= 1$$

but from Proposition 2.1.8

$$\omega(T) = r(T)$$
$$= r(S)$$
$$= \omega(S)$$
$$= 1$$

And again using Proposition 2.1.9

$$\| T \| = \omega(T)$$
$$= \omega(S)$$
$$= \| S \|$$
$$= 1$$

It is also evident that almost similarity in general does not preserve the norm of operators since almost similar does not in general imply similarity.

REFFERENCES

[1] Halmos, P. R. (1967). *A Hilbert space problem book*. Princeston, Van nostrand.

[2] Jibril, A. A. (1996). On Almost similar operators. *Arabian J. Sci. Engrg, 21*, 443-449.

[3] Musundi, S. W., Sitati, N. I., Nzimbi, B. M., & Murwayi, A. L. (2013). On almost similarity operator equivalence relation. *IJRRAS, 15*(3), 293-299.

[4] Nzimbi, B. M., Pokhariyal, G. P., & Khalaghai, J. M. (2008). A note on similarity, almost- similarity and equivalence of operators. *FJMS, 28*(2), 305-319.

[5] Skoufranis, P. (2014). Numerical ranges of operators. http://pskoufra.info.yorku .ca/files/2016/07/Numerical-Range.pdf. Accessed: 2017-08-01.