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Transitivity Action of the Cartesian Product of the Alternating Group Acting on a Cartesian Product of Ordered Sets of Triples

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this paper, we investigate some transitivity action properties of the cartesian product of the alternating group $A_n (n \geq 5)$ acting on a cartesian product of ordered sets of triples using the Orbit-Stabilizer Theorem by showing that the length of the orbit (p, s, v) in $A_n \times A_n \times A_n$, $(n \geq 5)$ acting on $P^{[3]} \times S^{[3]} \times V^{[3]}$ is equivalent to the cardinality of $P^{[3]} \times S^{[3]} \times V^{[3]}$ to imply transitivity.

Keywords: Orbits; stabilizer; transitivity action; ordered sets of triples; cartesian product; fixed point.

1 Preliminaries

1.1 Notation and Terminology

In this paper, we shall represent the following notations as: $\sum$- sum over i; A_n -an alternating group of degree n and order $\frac{n!}{2}$; $|G|$ – the order of a group G; $|G{:}H|$ -Index of H in G; $P^{[3]}$ – the set of an ordered triple from set

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$P = \{1,2,3,\dots,n\}$; $S^{[3]}$ – the set of an ordered triple from set $S = \{n+1, n+2,\dots,2n\}$; $V^{[3]}$ – the set of an ordered triple from set $V = \{2n+1, 2n+2,\dots,3n\}$; $[a,b,c]$ -Ordered triple; $A_n \times A_n \times A_n$ -Cartesian product of alternating group A_n; $P^{[3]} \times S^{[3]} \times V^{[3]}$ -Cartesian product of ordered sets of triples $P^{[3]}$, $S^{[3]}$ and $V^{[3]}$.

Definition 1.1.1. Group action [1]: Let P be a non-empty set. A group G is said to act on the left of P if for each $g \in G$ and each $p \in P$ there corresponds a unique element $gp \in G$ such that:

(i) $(g_1g_2)p = g_1(g_2p)$, $g_1, g_2 \in G$ and $p \in P$.
(ii) For any $p \in P$, $ep = p$, where e is the identity in G.

The action of G from the right on P can be defined in the same manner.

Definition 1.1.2. Orbit [2]: The action of a group G on a set P partitions P into disjoint equivalence classes referred to as orbits or transitivity classes of action. The orbit containing $p \in P$ is denoted by $Orb_G(p)$.

Definition 1.1.3. Stabilizer of an element [3]: Let G act on a set P and $p \in P$. The stabilizer of p in G, denoted by $Stab_G(p)$ is given by $Stab_G(p) = \{g \in G | gp = p\}$.

Definition 1.1.4. Fixed point [1]: Let G act on a set P. The set of elements of P fixed by $g \in G$ is called the fixed-point set of G and is denoted by $Fix(g)$. Thus $Fix(g) = \{p \in P | gp = p\}$.

Definition 1.1.5. Transitive group [4]: If the action of a group G on set P has only one orbit, then we say that G acts transitively on P. In other words, G acts transitively on P if for every pair of points $p, s \in P$, there exists $g \in G$ such that $gp = s$.

Definition 1.1.6. Conjugate group [2]: A group G with two subgroups H and K, then they are said to be conjugate if $H = gkg^{-1}$ for some $g \in G$.

Theorem 1.1.7 [5]: Two permutations in A_n are conjugate if and only if, they have the same cycle type and if $g \in S_n$ has cycle type $(\alpha_1, \alpha_2, \dots, \alpha_n)$, then the number of permutations in S_n conjugate to g is, $\frac{n!}{\prod_{i=1}^{n} \alpha_i! i^{\alpha i}}$.

Theorem 1.1.8 (Orbit – Stabilizer Theorem, [3, p.72]): Let G act on a set P. Then $|Orb_G(p)| = |G : Stab_G(p)|$.

Theorem 1.1.9 [3]: Let G be a group acting on a finite set P. Then the number of G-orbits in P is,

$$\frac{1}{|G|}\sum_{g\in G} |Fix\ (g)|.$$

Definition 1.1.10 (Direct product action, [4]): Let (G_1, P_1) and (G_2, P_2) be permutation groups. The direct product $G_1 \times G_2$ acts on the disjoint union $P_1 \cup P_2$ by the rule $p(g_1, g_2) = \begin{cases} pg_1, if\ p \in P_1, \\ pg_2, if\ p \in P_2 \end{cases}$ and on the Cartesian product $P_1 \times P_2$ by the rule $(p_1, p_2)(g_1, g_2) = (p_1g_1, p_2g_2)$.

Theorem 1.1.11 [6]: The $G_1 \times G_2 \times G_3$-orbit containing $(p, s, v) \in P \times S \times V$ is given by $Orb_{G_1}(p) \times Orb_{G_2}(s) \times Orb_{G_3}(v)$ and the stabilizer of (p, s, v) is given by $Stab_{G_1}(p) \times Stab_{G_2}(s) \times Stab_{G_3}(v)$.

2 Introduction

Higman [7] introduced the rank of a group on finite permutation groups of rank 3. In 1970, Higman proved that the rank of the symmetric group S_n acting on 2-element subsets from the set $P = \{1,2,\dots,n\}$ is 3 and the subdegrees are: 1 , $2(n-1)$ and $\binom{n-2}{2}$. Cameron [4] worked on the suborbits of multiply transitive permutations and later in 1974 studied the suborbits of primitive groups.

Ndarinyo *et al. [8]* showed that the alternating group $A_n = 5,6,7$ acts transitively on unordered and ordered triples from the set $P = 1,2,\dots,n$ when $n \leq 7$ through determination of the number of orbits. Nyaga [9] proved that the direct product action of the alternating group on the Cartesian product of three sets is transitive. The ranks and subdegrees associated with this action for $n \geq 4$ is 8; and $1, (\mathrm{n}-1), (\mathrm{n}-1)^2, (\mathrm{n}-1)^3$ respectively. Muriuki *et al.* [10] showed that for the action of direct product of three symmetric groups on Cartesian product of three sets, the action is both transitive and imprimitive for all $n \geq 2$ and the associated rank is 2^3. Mutua *et al.* [11] showed that the direct product of $S_n \times A_n$ on $X \times Y$ has its action both transitive and imprimitive when $n \geq 3$. The associated rank for this action is 6 when $n = 3$, but is 4 for all $n \geq 3$. Based on these results we investigate some properties of $A_n \times A_n \times A_n$, the cartesian product action of the alternating group acting on $P^{[3]} \times S^{[3]} \times V^{[3]}$, the cartesian product of ordered sets of triples.

The cartesian product of alternating group $A_n \times A_n \times A_n$, acts on $P^{[3]} \times S^{[3]} \times V^{[3]}$, by the rule;

$g_1\{([1,2,3],[1,2,4],\dots,[n,n-1,n-3],[n,n-1,n-2])\} \times g_2\{([n+1,n+2,n+3],[n+1,n+2,n+4],\dots,[2n,2n-1,2n-3],[2n,2n-1,2n-2])\} \times g_3\{([2n+1,2n+2,2n+3],[2n+1,2n+2,2n+4],\dots,[3n,3n-1,3n-3],[3n,3n-1,3n-3])\} = \{g_1([1,2,3],[1,2,4],\dots,[n,n-1,n-3],[n,n-1,n-2]), g_2([n+1,n+2,n+3],[n+1,n+2,n+4],\dots,[2n,2n-1,2n-3],[2n,2n-1,2n-2]), g_3([2n+1,2n+2,2n+3],[2n+1,2n+2,2n+4],\dots,[3n,3n-1,3n-3],[3n,3n-1,3n-3])\}$;

$\forall\, g_1, g_2, g_3 \in A_n, \{([1,2,3],[1,2,4],\dots,[n,n-1,n-3],[n,n-1,n-2])\} \in P^{[3]}$, set of ordered triples from the set $P = \{1,2,3,\dots,n\}$; $\{([n+1,n+2,n+3],[n+1,n+2,n+4],\dots,[2n,2n-1,2n-3],[2n,2n-1,2n-2])\} \in S^{[3]}$, set of ordered triples from the set $S = \{n+1,n+2,\dots,2n\}$; and $\{([2n+1,2n+2,2n+3],[2n+1,2n+2,2n+4],\dots,[3n,3n-1,3n-3],[3n,3n-1,3n-3])\} \in V^{[3]}$, set of ordered triples from the set $V = \{2n+1,2n+2,\dots,3n\}$.

3 Main Results

Lemma 2.1: The action of $\mathrm{A}_5 \times \mathrm{A}_5 \times \mathrm{A}_5$ on $P^{[3]} \times S^{[3]} \times V^{[3]}$ is transitive.

Proof: Let $G = A_5 \times A_5 \times A_5$ act on $P^{[3]} \times S^{[3]} \times V^{[3]}$ where; gap> Arrangements([1,2,3,4,5],3); $P^{[3]} =$ {[1, 2, 3], [1, 2, 4], [1, 2, 5], [1, 3, 2], [1, 3, 4], [1, 3, 5], [1, 4, 2], [1, 4, 3], [1, 4, 5], [1, 5, 2], [1, 5, 3], [1, 5, 4], [2, 1, 3], [2, 1, 4], [2, 1, 5], [2, 3, 1], [2, 3, 4], [2, 3, 5], [2, 4, 1], [2, 4, 3], [2, 4, 5], [2, 5, 1], [2, 5, 3], [2, 5, 4], [3, 1, 2], [3, 1, 4], [3, 1, 5], [3, 2, 1], [3, 2, 4], [3, 2, 5], [3, 4, 1], [3, 4, 2], [3, 4, 5], [3, 5, 1], [3, 5, 2], [3, 5, 4], [4, 1, 2], [4, 1, 3], [4, 1, 5], [4, 2, 1], [4, 2, 3], [4, 2, 5], [4, 3, 1], [4, 3, 2], [4, 3, 5], [4, 5, 1], [4, 5, 2], [4, 5, 3], [5, 1, 2], [5, 1, 3], [5, 1, 4], [5, 2, 1], [5, 2, 3], [5, 2, 4], [5, 3, 1], [5, 3, 2], [5, 3, 4], [5, 4, 1], [5, 4, 2], [5, 4, 3]} ;

gap> Arrangements([6,7,8,9,10],3); $S^{[3]}$ = { [6, 7, 8], [6, 7, 9], [6, 7, 10], [6, 8, 7], [6, 8, 9], [6, 8, 10], [6, 9, 7], [6, 9, 8], [6, 9, 10], [6, 10, 7],[6, 10, 8], [6, 10, 9], [7, 6, 8], [7, 6, 9], [7, 6, 10], [7, 8, 6], [7, 8, 9], [7, 8, 10], [7, 9, 6], [7, 9, 8], [7, 9, 10], [7, 10, 6], [7, 10, 8],[7, 10, 9], [8, 6, 7], [8, 6, 9], [8, 6, 10], [8, 7, 6], [8, 7, 9], [8, 7, 10], [8, 9, 6], [8, 9, 7], [8, 9, 10],[8, 10, 6], [8, 10, 7], [8, 10, 9], [9, 6, 7], [9, 6, 8], [9, 6, 10], [9, 7, 6], [9, 7, 8], [9, 7, 10], [9, 8, 6], [9, 8, 7], [9, 8, 10], [9, 10, 6],[9, 10, 7],

[9, 10, 8], [10, 6, 7], [10, 6, 8], [10, 6, 9], [10, 7, 6], [10, 7, 8], [10, 7, 9], [10, 8, 6], [10, 8, 7],[10, 8, 9], [10, 9, 6], [10, 9, 7], [10, 9, 8]} ; and

gap> Arrangements([11,12,13,14,15],3); $V^{[3]}$ = {[11, 12, 13],[11, 12, 14],[11, 12, 15], [11, 13, 12], [11, 13, 14],[11, 13, 15],[11, 14, 12], [11, 14, 13], [11, 14, 15], [11, 15, 12],[11, 15, 13], [11, 15, 14], [12, 11, 13], [12, 11, 14], [12, 11, 15], [12, 13, 11], [12, 13, 14], [12, 13, 15], [12, 14, 11], [12, 14, 13], [12, 14, 15],[12, 15, 11], [12, 15, 13],[12, 15, 14], [13, 11, 12], [13, 11, 14], [13, 11, 15], [13, 12, 11], [13, 12, 14], [13, 12, 15], [13, 14, 11],[13, 14, 12], [13, 14, 15], [13, 15, 11], [13, 15, 12], [13, 15, 14], [14, 11, 12], [14, 11, 13], [14, 11, 15], [14, 12, 11], [14, 12, 13], [14, 12, 15],[14, 13, 11], [14, 13, 12], [14, 13, 15],[14, 15, 11], [14, 15, 12], [14, 15, 13], [15, 11, 12], [15, 11, 13], [15, 11, 14], [15, 12, 11],[15, 12, 13], [15, 12, 14], [15, 13, 11], [15, 13, 12], [15, 13, 14], [15, 14, 11], [15, 14, 12], [15, 14, 13]} . The cartesian product of $P^{[3]} \times S^{[3]} \times V^{[3]}$ is generated using the GAP software with, $\left|P^{[3]} \times S^{[3]} \times V^{[3]}\right| = 216\ 000$. G is generated by

$< \{(1\ 2\ 3\ 4\ 5), (123)\}, \{(6\ 7\ 8\ 9\ 10), (678)\}, \{(11\ 12\ 13\ 14\ 15), (11\ 12\ 13)\} >$ using the GAP software. $([1,2,3], [6,7,8], [11,12,13])$ is fixed by an element $(g_p, g_s, g_v) \in G$ if and only if 1,2 and 3 comes from a single cycle in g_p ; 6,7 and 8 comes from a single cycle in g_s and 11,12 and 13 comes from a single cycle in g_v .

Therefore, $Stab_G([1,2,3], [6,7,8], [11,12,13]) = \{1,6,11\} = \{(e_p, e_s, e_v)\}$. $|Stab_G([1,2,3], [6,7,8], [11,12,13])| = 1$.

By Orbit-Stabilizer Theorem,

$$|Orb_G\ ([1,2,3], [6,7,8], [11,12,13])| = |G: Stab_G([1,2,3], [6,7,8], [11,12,13])|$$

$$= \frac{|G|}{|Stab_G([1,2,3], [6,7,8], [11,12,13])|}$$

$$= \frac{216000}{1} = 216000 = \left|P^{[3]} \times S^{[3]} \times V^{[3]}\right|$$

Therefore, $A_5 \times A_5 \times A_5$ acts transitively on $P^{[3]} \times S^{[3]} \times V^{[3]}$.

Lemma 2.2: The action of $A_6 \times A_6 \times A_6$ on $P^{[3]} \times S^{[3]} \times V^{[3]}$ is transitive.

Proof: Let $G = A_6 \times A_6 \times A_6$ act on $P^{[3]} \times S^{[3]} \times V^{[3]}$ where;

gap> Arrangements([1,2,3,4,5,6],3); $P^{[3]}$ =[1, 2, 3], [1, 2, 4], [1, 2, 5], [1, 2, 6], [1, 3, 2], [1, 3, 4], [1, 3, 5], [1, 3, 6], [1, 4, 2], [1, 4, 3], [1, 4, 5], [1, 4, 6], [1, 5, 2], [1, 5, 3], [1, 5, 4], [1, 5, 6],
[1, 6, 2], [1, 6, 3], [1, 6, 4], [1, 6, 5], [2, 1, 3], [2, 1, 4], [2, 1, 5], [2, 1, 6],
[2, 3, 1], [2, 3, 4], [2, 3, 5], [2, 3, 6], [2, 4, 1], [2, 4, 3], [2, 4, 5], [2, 4, 6],
[2, 5, 1], [2, 5, 3], [2, 5, 4], [2, 5, 6], [2, 6, 1], [2, 6, 3], [2, 6, 4], [2, 6, 5],
[3, 1, 2], [3, 1, 4], [3, 1, 5], [3, 1, 6], [3, 2, 1], [3, 2, 4], [3, 2, 5], [3, 2, 6],
[3, 4, 1], [3, 4, 2], [3, 4, 5], [3, 4, 6], [3, 5, 1], [3, 5, 2], [3, 5, 4], [3, 5, 6],
[3, 6, 1], [3, 6, 2], [3, 6, 4], [3, 6, 5], [4, 1, 2], [4, 1, 3], [4, 1, 5], [4, 1, 6],
[4, 2, 1], [4, 2, 3], [4, 2, 5], [4, 2, 6], [4, 3, 1], [4, 3, 2], [4, 3, 5], [4, 3, 6],
[4, 5, 1], [4, 5, 2], [4, 5, 3], [4, 5, 6], [4, 6, 1], [4, 6, 2], [4, 6, 3], [4, 6, 5],
[5, 1, 2], [5, 1, 3], [5, 1, 4], [5, 1, 6], [5, 2, 1], [5, 2, 3], [5, 2, 4], [5, 2, 6],
[5, 3, 1], [5, 3, 2], [5, 3, 4], [5, 3, 6], [5, 4, 1], [5, 4, 2], [5, 4, 3], [5, 4, 6],
[5, 6, 1], [5, 6, 2], [5, 6, 3], [5, 6, 4], [6, 1, 2], [6, 1, 3], [6, 1, 4], [6, 1, 5],

[6, 2, 1], [6, 2, 3], [6, 2, 4], [6, 2, 5], [6, 3, 1], [6, 3, 2], [6, 3, 4], [6, 3, 5],
[6, 4, 1], [6, 4, 2], [6, 4, 3], [6, 4, 5], [6, 5, 1], [6, 5, 2], [6, 5, 3], [6, 5, 4]};

gap> Arrangements([7,8,9,10,11,12],3); $S^{[3]}$ ={[7, 8, 9], [7, 8, 10], [7, 8, 11], [7, 8, 12], [7, 9, 8], [7, 9, 10], [7, 9, 11], [7, 9, 12], [7, 10, 8], [7, 10, 9],[7, 10, 11], [7, 10, 12], [7, 11, 8], [7, 11, 9], [7, 11, 10], [7, 11, 12], [7, 12, 8], [7, 12, 9], [7, 12, 10], [7, 12, 11], [8, 7, 9], [8, 7, 10], [8, 7, 11], [8, 7, 12], [8, 9, 7], [8, 9, 10], [8, 9, 11], [8, 9, 12], [8, 10, 7], [8, 10, 9], [8, 10, 11], [8, 10, 12], [8, 11, 7],[8, 11, 9], [8, 11, 10], [8, 11, 12], [8, 12, 7], [8, 12, 9], [8, 12, 10], [8, 12, 11], [9, 7, 8], [9, 7, 10], [9, 7, 11], [9, 7, 12], [9, 8, 7], [9, 8, 10],[9, 8, 11], [9, 8, 12], [9, 10, 7], [9, 10, 8], [9, 10, 11], [9, 10, 12], [9, 11, 7], [9, 11, 8], [9, 11, 10], [9, 11, 12], [9, 12, 7], [9, 12, 8], [9, 12, 10], [9, 12, 11], [10, 7, 8], [10, 7, 9], [10, 7, 11], [10, 7, 12], [10, 8, 7], [10, 8, 9], [10, 8, 11], [10, 8, 12], [10, 9, 7], [10, 9, 8], [10, 9, 11], [10, 9, 12], [10, 11, 7], [10, 11, 8], [10, 11, 9], [10, 11, 12], [10, 12, 7], [10, 12, 8], [10, 12, 9], [10, 12, 11], [11, 7, 8], [11, 7, 9], [11, 7, 10], [11, 7, 12], [11, 8, 7], [11, 8, 9], [11, 8, 10], [11, 8, 12], [11, 9, 7], [11, 9, 8], [11, 9, 10], [11, 9, 12], [11, 10, 7], [11, 10, 8], [11, 10, 9], [11, 10, 12], [11, 12, 7], [11, 12, 8],[11, 12, 9], [11, 12, 10], [12, 7, 8], [12, 7, 9], [12, 7, 10], [12, 7, 11], [12, 8, 7], [12, 8, 9], [12, 8, 10], [12, 8, 11], [12, 9, 7], [12, 9, 8], [12, 9, 10], [12, 9, 11], [12, 10, 7], [12, 10, 8], [12, 10, 9], [12, 10, 11], [12, 11, 7], [12, 11, 8], [12, 11, 9], [12, 11, 10]}
and;

gap> Arrangements([13,14,15,16,17,18],3); $V^{[3]}$ ={[13, 14, 15], [13, 14, 16], [13, 14, 17], [13, 14, 18], [13, 15, 14], [13, 15, 16], [13, 15, 17], [13, 15, 18], [13, 16, 14], [13, 16, 15],[13, 16, 17], [13, 16, 18], [13, 17, 14], [13, 17, 15], [13, 17, 16], [13, 17, 18], [13, 18, 14], [13, 18, 15], [13, 18, 16], [13, 18, 17], [14, 13, 15], [14, 13, 16], [14, 13, 17], [14, 13, 18], [14, 15, 13], [14, 15, 16], [14, 15, 17], [14, 15, 18], [14, 16, 13], [14, 16, 15], [14, 16, 17], [14, 16, 18], [14, 17, 13], [14, 17, 15], [14, 17, 16], [14, 17, 18], [14, 18, 13], [14, 18, 15], [14, 18, 16], [14, 18, 17], [15, 13, 14], [15, 13, 16], [15, 13, 17], [15, 13, 18], [15, 14, 13], [15, 14, 16], [15, 14, 17], [15, 14, 18], [15, 16, 13], [15, 16, 14], [15, 16, 17], [15, 16, 18], [15, 17, 13], [15, 17, 14], [15, 17, 16], [15, 17, 18], [15, 18, 13], [15, 18, 14], [15, 18, 16], [15, 18, 17], [16, 13, 14], [16, 13, 15], [16, 13, 17], [16, 13, 18], [16, 14, 13], [16, 14, 15], [16, 14, 17], [16, 14, 18], [16, 15, 13], [16, 15, 14], [16, 15, 17], [16, 15, 18], [16, 17, 13], [16, 17, 14], [16, 17, 15], [16, 17, 18], [16, 18, 13], [16, 18, 14], [16, 18, 15], [16, 18, 17], [17, 13, 14], [17, 13, 15], [17, 13, 16], [17, 13, 18], [17, 14, 13], [17, 14, 15], [17, 14, 16], [17, 14, 18], [17, 15, 13], [17, 15, 14], [17, 15, 16], [17, 15, 18], [17, 16, 13], [17, 16, 14], [17, 16, 15], [17, 16, 18], [17, 18, 13], [17, 18, 14], [17, 18, 15], [17, 18, 16], [18, 13, 14], [18, 13, 15], [18, 13, 16], [18, 13, 17], [18, 14, 13], [18, 14, 15], [18, 14, 16], [18, 14, 17], [18, 15, 13], [18, 15, 14], [18, 15, 16], [18, 15, 17], [18, 16, 13], [18, 16, 14], [18, 16, 15], [18, 16, 17], [18, 17, 13], [18, 17, 14], [18, 17, 15], [18, 17, 16]}.

The cartesian product of $P^{[3]} \times S^{[3]} \times V^{[3]}$ is generated using the GAP software with $\left|P^{[3]} \times S^{[3]} \times V^{[3]}\right| = 1728000$. G is generated by $< \{(123456), (123)\}, \{(7\ 8\ 9\ 10\ 11\ 12), (789)\}, \{(13\ 14\ 15\ 16\ 17\ 18), (13\ 14\ 15)\} >$ using the GAP software. $([1,2,3], [7,8,9], [13,14,15])$ is fixed by an element $(g_p, g_s, g_v) \in G$ if and only if 1,2 and 3 comes from a single cycle in g_p ; 7,8 and 9 comes from a single cycle in g_s and 13,14 and 15 comes from a single cycle of g_v .

The $|Stab_G([1,2,3], [7,8,9], [13,14,15])| = 27$.

By Orbit-Stabilizer Theorem,

$$|Orb_G([1,2,3], [7,8,9], [13,14,15])| = |G: Stab_G([1,2,3], [7,8,9], [13,14,15])|$$

$$= \frac{|G|}{|Stab_G([1,2,3], [7,8,9], [13,14,15])|}$$

$$= \frac{46\ 656\ 000}{27} = 1728000 = \left|P^{[3]} \times S^{[3]} \times V^{[3]}\right|$$

Therefore, $A_6 \times A_6 \times A_6$ acts transitively on $P^{[3]} \times S^{[3]} \times V^{[3]}$.

Lemma 2.3: The action of $A_7 \times A_7 \times A_7$ on $P^{[3]} \times S^{[3]} \times V^{[3]}$ is transitive.

Proof: Let $G = A_7 \times A_7 \times A_7$ act on $P^{[3]} \times S^{[3]} \times V^{[3]}$ where;

gap> Arrangements([1,2,3,4,5,6,7],3); $P^{[3]} =$ {[1, 2, 3], [1, 2, 4], [1, 2, 5], [1, 2, 6], [1, 2, 7], [1, 3, 2], [1, 3, 4], [1, 3, 5], [1, 3, 6], [1, 3, 7], [1, 4, 2], [1, 4, 3], [1, 4, 5], [1, 4, 6], [1, 4, 7], [1, 5, 2], [1, 5, 3], [1, 5, 4], [1, 5, 6], [1, 5, 7], [1, 6, 2], [1, 6, 3], [1, 6, 4], [1, 6, 5], [1, 6, 7], [1, 7, 2], [1, 7, 3], [1, 7, 4], [1, 7, 5], [1, 7, 6], [2, 1, 3], [2, 1, 4], [2, 1, 5], [2, 1, 6], [2, 1, 7], [2, 3, 1], [2, 3, 4], [2, 3, 5], [2, 3, 6], [2, 3, 7], [2, 4, 1], [2, 4, 3], [2, 4, 5], [2, 4, 6], [2, 4, 7], [2, 5, 1], [2, 5, 3], [2, 5, 4], [2, 5, 6], [2, 5, 7], [2, 6, 1], [2, 6, 3], [2, 6, 4], [2, 6, 5], [2, 6, 7], [2, 7, 1], [2, 7, 3], [2, 7, 4], [2, 7, 5], [2, 7, 6], [3, 1, 2], [3, 1, 4], [3, 1, 5], [3, 1, 6], [3, 1, 7], [3, 2, 1], [3, 2, 4], [3, 2, 5], [3, 2, 6], [3, 2, 7], [3, 4, 1], [3, 4, 2], [3, 4, 5], [3, 4, 6], [3, 4, 7], [3, 5, 1], [3, 5, 2], [3, 5, 4], [3, 5, 6], [3, 5, 7], [3, 6, 1], [3, 6, 2], [3, 6, 4], [3, 6, 5], [3, 6, 7], [3, 7, 1], [3, 7, 2], [3, 7, 4], [3, 7, 5], [3, 7, 6], [4, 1, 2], [4, 1, 3], [4, 1, 5], [4, 1, 6], [4, 1, 7], [4, 2, 1],[4, 2, 3], [4, 2, 5], [4, 2, 6], [4, 2, 7], [4, 3, 1], [4, 3, 2], [4, 3, 5], [4, 3, 6], [4, 3, 7], [4, 5, 1], [4, 5, 2], [4, 5, 3], [4, 5, 6], [4, 5, 7], [4, 6, 1], [4, 6, 2], [4, 6, 3], [4, 6, 5], [4, 6, 7], [4, 7, 1], [4, 7, 2], [4, 7, 3], [4, 7, 5], [4, 7, 6], [5, 1, 2], [5, 1, 3], [5, 1, 4], [5, 1, 6], [5, 1, 7], [5, 2, 1], [5, 2, 3], [5, 2, 4], [5, 2, 6], [5, 2, 7], [5, 3, 1], [5, 3, 2], [5, 3, 4], [5, 3, 6], [5, 3, 7], [5, 4, 1], [5, 4, 2], [5, 4, 3], [5, 4, 6], [5, 4, 7], [5, 6, 1], [5, 6, 2], [5, 6, 3], [5, 6, 4], [5, 6, 7], [5, 7, 1], [5, 7, 2], [5, 7, 3], [5, 7, 4], [5, 7, 6], [6, 1, 2], [6, 1, 3], [6, 1, 4], [6, 1, 5], [6, 1, 7], [6, 2, 1], [6, 2, 3], [6, 2, 4], [6, 2, 5], [6, 2, 7], [6, 3, 1], [6, 3, 2], [6, 3, 4], [6, 3, 5], [6, 3, 7], [6, 4, 1], [6, 4, 2], [6, 4, 3], [6, 4, 5], [6, 4, 7], [6, 5, 1], [6, 5, 2], [6, 5, 3], [6, 5, 4], [6, 5, 7], [6, 7, 1], [6, 7, 2], [6, 7, 3], [6, 7, 4], [6, 7, 5], [7, 1, 2], [7, 1, 3], [7, 1, 4], [7, 1, 5], [7, 1, 6], [7, 2, 1], [7, 2, 3], [7, 2, 4], [7, 2, 5], [7, 2, 6], [7, 3, 1], [7, 3, 2], [7, 3, 4], [7, 3, 5], [7, 3, 6], [7, 4, 1], [7, 4, 2], [7, 4, 3], [7, 4, 5], [7, 4, 6], [7, 5, 1], [7, 5, 2], [7, 5, 3], [7, 5, 4], [7, 5, 6], [7, 6, 1], [7, 6, 2], [7, 6, 3], [7, 6, 4], [7, 6, 5]};

gap> Arrangements([8,9,10,11,12,13,14],3); $S^{[3]} =$ {[8, 9, 10], [8, 9, 11], [8, 9, 12], [8, 9, 13], [8, 9, 14], [8, 10, 9], [8, 10, 11], [8, 10, 12], [8, 10, 13], [8, 10, 14], [8, 11, 9], [8, 11, 10], [8, 11, 12], [8, 11, 13], [8, 11, 14], [8, 12, 9], [8, 12, 10], [8, 12, 11], [8, 12, 13], [8, 12, 14], [8, 13, 9], [8, 13, 10], [8, 13, 11], [8, 13, 12], [8, 13, 14], [8, 14, 9], [8, 14, 10], [8, 14, 11], [8, 14, 12], [8, 14, 13], [9, 8, 10], [9, 8, 11], [9, 8, 12], [9, 8, 13], [9, 8, 14], [9, 10, 8], [9, 10, 11], [9, 10, 12], [9, 10, 13], [9, 10, 14], [9, 11, 8], [9, 11, 10], [9, 11, 12], [9, 11, 13], [9, 11, 14], [9, 12, 8], [9, 12, 10], [9, 12, 11], [9, 12, 13], [9, 12, 14], [9, 13, 8], [9, 13, 10], [9, 13, 11], [9, 13, 12], [9, 13, 14], [9, 14, 8], [9, 14, 10], [9, 14, 11], [9, 14, 12], [9, 14, 13], [10, 8, 9], [10, 8, 11], [10, 8, 12], [10, 8, 13], [10, 8, 14], [10, 9, 8],[10, 9, 11], [10, 9, 12], [10, 9, 13], [10, 9, 14], [10, 11, 8], [10, 11, 9], [10, 11, 12], [10, 11, 13], [10, 11, 14], [10, 12, 8], [10, 12, 9], [10, 12, 11], [10, 12, 13], [10, 12, 14], [10, 13, 8], [10, 13, 9], [10, 13, 11], [10, 13, 12], [10, 13, 14], [10, 14, 8], [10, 14, 9], [10, 14, 11], [10, 14, 12], [10, 14, 13], [11, 8, 9], [11, 8, 10], [11, 8, 12], [11, 8, 13], [11, 8, 14], [11, 9, 8], [11, 9, 10], [11, 9, 12], [11, 9, 13], [11, 9, 14], [11, 10, 8], [11, 10, 9], [11, 10, 12], [11, 10, 13], [11, 10, 14], [11, 12, 8], [11, 12, 9], [11, 12, 10], [11, 12, 13], [11, 12, 14], [11, 13, 8], [11, 13, 9], [11, 13, 10], [11, 13, 12], [11, 13, 14], [11, 14, 8], [11, 14, 9], [11, 14, 10], [11, 14, 12], [11, 14, 13], [12, 8, 9], [12, 8, 10], [12, 8, 11], [12, 8, 13], [12, 8, 14], [12, 9, 8], [12, 9, 10], [12, 9, 11], [12, 9, 13], [12, 9, 14], [12, 10, 8], [12, 10, 9],[12, 10, 11], [12, 10, 13], [12, 10, 14], [12, 11, 8], [12, 11, 9], [12, 11, 10], [12, 11, 13], [12, 11, 14], [12, 13, 8], [12, 13, 9], [12, 13, 10], [12, 13, 11], [12, 13, 14], [12, 14, 8], [12, 14, 9], [12, 14, 10], [12, 14, 11], [12, 14, 13], [13, 8, 9], [13, 8, 10], [13, 8, 11], [13, 8, 12], [13, 8, 14], [13, 9, 8], [13, 9, 10], [13, 9, 11], [13, 9, 12], [13, 9, 14], [13, 10, 8], [13, 10, 9], [13, 10, 11], [13, 10, 12],[13, 10, 14], [13, 11, 8], [13, 11, 9], [13, 11, 10], [13, 11, 12], [13, 11, 14], [13, 12, 8], [13, 12, 9], [13, 12, 10], [13, 12, 11], [13, 12, 14], [13, 14, 8], [13, 14, 9], [13, 14, 10], [13, 14, 11], [13, 14, 12], [14, 8, 9], [14, 8, 10], [14, 8, 11], [14, 8, 12], [14, 8, 13], [14, 9, 8], [14, 9, 10], [14, 9, 11], [14, 9, 12], [14, 9, 13], [14, 10, 8], [14, 10, 9], [14, 10, 11], [14, 10, 12], [14, 10, 13], [14, 11, 8], [14, 11, 9], [14, 11, 10], [14, 11, 12], [14, 11, 13], [14, 12, 8], [14, 12, 9], [14, 12, 10], [14, 12, 11], [14, 12, 13], [14, 13, 8], [14, 13, 9], [14, 13, 10], [14, 13, 11], [14, 13, 12]};

and

gap> Arrangements([15,16,17,18,19,20,21],3); $V^{[3]}$ ={[15, 16, 17], [15, 16, 18], [15, 16, 19], [15, 16, 20], [15, 16, 21], [15, 17, 16], [15, 17, 18], [15, 17, 19], [15, 17, 20], [15, 17, 21], [15, 18, 16], [15, 18, 17], [15, 18, 19], [15, 18, 20], [15, 18, 21], [15, 19, 16], [15, 19, 17], [15, 19, 18], [15, 19, 20], [15, 19, 21], [15, 20, 16], [15, 20, 17], [15, 20, 18], [15, 20, 19], [15, 20, 21], [15, 21, 16], [15, 21, 17], [15, 21, 18], [15, 21, 19], [15, 21, 20], [16, 15, 17], [16, 15, 18], [16, 15, 19], [16, 15, 20], [16, 15, 21], [16, 17, 15], [16, 17, 18], [16, 17, 19], [16, 17, 20], [16, 17, 21], [16, 18, 15], [16, 18, 17], [16, 18, 19], [16, 18, 20], [16, 18, 21], [16, 19, 15], [16, 19, 17], [16, 19, 18], [16, 19, 20], [16, 19, 21], [16, 20, 15], [16, 20, 17], [16, 20, 18], [16, 20, 19], [16, 20, 21], [16, 21, 15], [16, 21, 17], [16, 21, 18], [16, 21, 19], [16, 21, 20], [17, 15, 16], [17, 15, 18], [17, 15, 19], [17, 15, 20], [17, 15, 21], [17, 16, 15], [17, 16, 18], [17, 16, 19], [17, 16, 20], [17, 16, 21], [17, 18, 15], [17, 18, 16], [17, 18, 19], [17, 18, 20], [17, 18, 21], [17, 19, 15], [17, 19, 16], [17, 19, 18], [17, 19, 20], [17, 19, 21], [17, 20, 15], [17, 20, 16], [17, 20, 18], [17, 20, 19], [17, 20, 21], [17, 21, 15], [17, 21, 16], [17, 21, 18], [17, 21, 19], [17, 21, 20], [18, 15, 16], [18, 15, 17], [18, 15, 19], [18, 15, 20], [18, 15, 21], [18, 16, 15], [18, 16, 17], [18, 16, 19], [18, 16, 20], [18, 16, 21], [18, 17, 15], [18, 17, 16], [18, 17, 19], [18, 17, 20], [18, 17, 21], [18, 19, 15], [18, 19, 16], [18, 19, 17], [18, 19, 20], [18, 19, 21], [18, 20, 15], [18, 20, 16], [18, 20, 17], [18, 20, 19], [18, 20, 21], [18, 21, 15], [18, 21, 16], [18, 21, 17], [18, 21, 19], [18, 21, 20], [19, 15, 16], [19, 15, 17], [19, 15, 18], [19, 15, 20], [19, 15, 21], [19, 16, 15], [19, 16, 17], [19, 16, 18], [19, 16, 20], [19, 16, 21], [19, 17, 15], [19, 17, 16], [19, 17, 18], [19, 17, 20], [19, 17, 21], [19, 18, 15], [19, 18, 16], [19, 18, 17], [19, 18, 20], [19, 18, 21], [19, 20, 15], [19, 20, 16], [19, 20, 17], [19, 20, 18], [19, 20, 21], [19, 21, 15], [19, 21, 16], [19, 21, 17], [19, 21, 18], [19, 21, 20], [20, 15, 16], [20, 15, 17], [20, 15, 18], [20, 15, 19], [20, 15, 21], [20, 16, 15], [20, 16, 17], [20, 16, 18], [20, 16, 19], [20, 16, 21], [20, 17, 15], [20, 17, 16], [20, 17, 18], [20, 17, 19], [20, 17, 21], [20, 18, 15], [20, 18, 16], [20, 18, 17], [20, 18, 19], [20, 18, 21], [20, 19, 15], [20, 19, 16], [20, 19, 17], [20, 19, 18], [20, 19, 21], [20, 21, 15], [20, 21, 16], [20, 21, 17], [20, 21, 18], [20, 21, 19], [21, 15, 16], [21, 15, 17], [21, 15, 18], [21, 15, 19], [21, 15, 20], [21, 16, 15], [21, 16, 17], [21, 16, 18], [21, 16, 19], [21, 16, 20], [21, 17, 15], [21, 17, 16], [21, 17, 18], [21, 17, 19], [21, 17, 20], [21, 18, 15], [21, 18, 16], [21, 18, 17], [21, 18, 19], [21, 18, 20], [21, 19, 15], [21, 19, 16], [21, 19, 17], [21, 19, 18], [21, 19, 20], [21, 20, 15], [21, 20, 16], [21, 20, 17], [21, 20, 18], [21, 20, 19]}.

The cartesian product of $P^{[3]} \times S^{[3]} \times V^{[3]}$ is generated using the GAP software with, $\left|P^{[3]} \times S^{[3]} \times V^{[3]}\right| = 9\,261\,000$. G is generated by

$< \{(1234567),(123)\},\{(8\ 9\ 10\ 11\ 12\ 13\ 14),(8\ 9\ 10)\},\{(15\ 16\ 17\ 18\ 19\ 20\ 21),(15\ 16\ 17)\} >$ using the GAP software. $([1,2,3],[8,9,10],[15,16,17])$ is fixed by an element $(g_p, g_s, g_v) \in G$ if and only if *1,2* and *3* comes from a single cycle of g_p ; *8,9* and *10* comes from a single cycle of g_s and *15,16* and *17* comes from a single cycle of g_v .

The $\left|Stab_G\left([1,2,3],[8,9,10],[15,16,17]\right)\right| = 1728$.

By Orbit-Stabilizer Theorem,

$$\left|Orb_G\left([1,2,3],[8,9,10],[15,16,17]\right)\right| = \left|G: Stab_G\left([1,2,3],[8,9,10],[15,16,17]\right)\right|$$

$$= \frac{|G|}{\left|Stab_G\left([1,2,3],[8,9,10],[15,16,17]\right)\right|}$$

$$= \frac{16\,003\,008\,000}{1728} = 9261000 = \left|P^{[3]} \times S^{[3]} \times V^{[3]}\right|$$

Therefore, $A_7 \times A_7 \times A_7$ acts transitively on $P^{[3]} \times S^{[3]} \times V^{[3]}$.

Theorem 2.4: The action of $A_n \times A_n \times A_n$ on $P^{[3]} \times S^{[3]} \times V^{[3]}$ is transitive if and only if $n \geq 5$.

Proof: Let $G = G_p \times G_s \times G_v = A_n \times A_n \times A_n$ act on $P^{[3]} \times S^{[3]} \times V^{[3]}$. It suffices to verify that $\left|P^{[3]} \times S^{[3]} \times V^{[3]}\right|$ is equal to $\left|Orb_G\left([1,2,3],[n+1,n+2,n+3],[2n+1,2n+2,2n+3]\right)\right|$.

Let $|R| = \left|Stab_G\left([1,2,3],[n+1,n+2,n+3],[2n+1,2n+2,2n+3]\right)\right|$.

So, $\left(g_p, g_s, g_v\right) \in G = A_n \times A_n \times A_n$ fixes $\left([1,2,3],[n+1,n+2,n+3],[2n+1,2n+2,2n+3]\right) \in P^{[3]} \times S^{[3]} \times V^{[3]}$ if and only if *1,2* and *3* comes from *1*-cycle of g_p ; $n+1, n+2$ and $n+3$ comes from *1*-cycle of g_s and $2n+1, 2n+2$ and $2n+3$comes from *1*-cycle of g_v .

The $Stab_G\left([1,2,3],[n+1,n+2,n+3],[2n+1,2n+2,2n+3]\right)$ is isomorphic to: $A_{n-3} \times A_{n-3} \times A_{n-3}$.

Therefore, $|R| = \left|Stab_G\left([1,2,3],[n+1,n+2,n+3],[2n+1,2n+2,2n+3]\right)\right| = \left|Stab_{G_p}\left([1,2,3]\right) \times Stab_{G_s}\left([n+1,n+2,n+3]\right) \times Stab_{G_v}\left([2n+1,2n+2,2n+3]\right)\right|$

$$|R| = \frac{(n-3)! \times (n-3)! \times (n-3)!}{2 \times 2 \times 2} = \left(\frac{(n-3)!}{2}\right)^3$$

Applying the Orbit-Stabilizer Theorem we get;

$$\left|Orb_G\left([1,2,3],[n+1,n+2,n+3],[2n+1,2n+2,2n+3]\right)\right|$$

$$= \left|G : Stab_G\left([1,2,3],[n+1,n+2,n+3],[2n+1,2n+2,2n+3]\right)\right|$$

$$|G| = \frac{n! \times n! \times n!}{2 \times 2 \times 2} = \left(\frac{n!}{2}\right)^3$$

$$\frac{|G|}{|R|} = \frac{\left(\frac{n!}{2}\right)^3}{\left(\frac{(n-3)!}{2}\right)^3} = \left(\frac{n!}{(n-3)!}\right)^3 .$$

Therefore;

$$\frac{|G|}{|R|} = \left(\frac{n!}{(n-3)!}\right)^3 = \left|P^{[3]} \times S^{[3]} \times V^{[3]}\right|$$

Hence, $A_n \times A_n \times A_n$ acts transitively on $P^{[3]} \times S^{[3]} \times V^{[3]}$ if $n \geq 5$.

Corollary 2.5: For $n < 5$, the

$\left|Stab_G\left([1,2,3],[n+1,n+2,n+3],[2n+1,2n+2,2n+3]\right)\right| = \left|A_{n-3} \times A_{n-3} \times A_{n-3}\right| < 1.$

4. Conclusion

The cartesian product of the alternating group $A\_n(n \geq 5)$ acting on a cartesian product of ordered sets of triples has been determined to be transitive using the Orbit-Stabilizer Theorem by showing that the length of the orbit (p, s, v) in $A_n \times A_n \times A_n$, $(n \geq 5)$ acting on $P^{[3]} \times S^{[3]} \times V^{[3]}$ is equivalent to the cardinality of $P^{[3]} \times S^{[3]} \times V^{[3]}$ to imply transitivity.

Disclaimer

The products used for this research are commonly and predominantly use products in our area of research and country. There is absolutely no conflict of interest between the authors and producers of the products because we do not intend to use these products as an avenue for any litigation but for the advancement of knowledge. Also, the research was not funded by the producing company rather it was funded by personal efforts of the authors.

Competing Interests

Authors have declared that no competing interests exist.

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