# Correlation Between Electromagnetic Wave Equation And Einstein Theory Of Relativity In Derivation Of Schrödinger Equation And Hilbert Space Operators

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***Abstract* — Operators in Hilbert space have properties which are useful in the study of mathematical abstract areas such as approximation theory, Banach Fixed point theory, the spectral theory as well as Quantum Mechanics. Schrödinger equation is a fundamental entity with many applications in Quantum Mechanics. This equation was initially derived by applying the knowledge of electromagnetic wave function and Einstein theory of relativity. Later, it was derived by applying the knowledge of Newtonian mechanics. It was also derived by extending the wave equation for classical fields to photons and simplified using approximations consistent with generalized non-zero rest mass. However, from the existing literature no study has been done on deriving Schrödinger equation using properties of Hilbert space operators. In this study, Hilbert space operators that include unitary operators, self adjoint operators and compact operators, norms of linear operators, Hilbert Schmidt operator, normal operators together with Lebesque Integral, Neumann Integral and spectrum are used in place of the existing concepts of electromagnetic wave function, Einstein theory of relativity and approximation consistent with generalized non zero mass to derive the Schrödinger equation. The derivation of Schrödinger equation and its application using Hilbert space operators enhances a better understanding of the concept of Schrödinger equation. The results of this work can further find use in quantum mechanics as well as in mathematical operator theory.**

***Keywords— Hilbert Space Operators; Electromagnetic wave function; Einstein theory of relativity and Schrodinger equation;***

## I. Introduction

Schrödinger equation was first derived by Schrödinger in 1926. In his work he used the knowledge of electromagnetic prototype of wave equation $(v^2\nabla^2 - \frac{d^2}{dt^2})E$ and Einstein theory of relativity $(E = mc^2)$ [20]. The purpose of his study was to find the wave function of the electron. [11] used Newtonian mechanics to derive Schrödinger equation. In his work he used the hypothesis that any particle of mass *m* constantly undergoes Brownian motion with diffusion co-efficient $\frac{\hbar}{2m}$. [20] derived Schrödinger equation by extending the wave equation for classical fields to photons and generalized to non-zero rest mass particles using approximations consistent with non-relativistic particles.

Hilbert space gives a means by which one can consider functions as points belonging to an infinite dimensional space. In [10], the states of quantum systems are identified by unit vectors in an infinite dimensional complex Hilbert space and observables such as position, momentum and energy are realized as self-adjoint linear operators acting on the space. Consequently, [10] showed the relationship between the needs of physics and the mathematics of operators in Hilbert spaces.

According to [15], the concept of Hilbert spaces was first introduced by David Hilbert between 1862- 1943. These Hilbert spaces are complete inner product spaces.

If $T$ is an operator on Hilbert space $\mathcal{H}$ then:

i. $T$ is normal if $TT^* = T^*T$.
ii. $T$ is self-adjoint (or Hermitian) if $T = T^*$.
iii. $T$ is positive if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$.
iv. $T$ is unitary if $TT^* = T^*T = 1$.

As per [7], for an operator $T \in B(\mathcal{H})$ on Hilbert space, and by Reisz Representation theorem, there exists a unique vector $z = z_y \in \mathcal{H}$ so that $\langle y, Tx\rangle = \langle z_y, x\rangle$ for all $x \in \mathcal{H}$. The map $T^*: \mathcal{H} \to \mathcal{H}$ is defined as $T^*y = z_y$. By construction

$\langle T^*y, x\rangle = \langle y, Tx\rangle\ \forall x, y \in \mathcal{H}$ , the condition uniquely determines $T^*y$ for $y \in \mathcal{H}$ . Thus $T^*$ is an adjoint operator of $T$.

If two elements of the set $M$ are pairwise orthogonal vectors, each of the vector is normalized and each has a norm equal to one, then the set $M$ is called orthonormal [1].

The definition of Riemann's integral is adopted from [10]. Let $f$ be defined on $[a, b]$, then $f$ is said to be *Riemann integrable* on $[a, b]$ if there is a number $L$ with the following property. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\sigma - L| < \varepsilon$. If $\sigma$ is Riemann's sum of $f$ over partition $P$ of $[a, b]$ such that $\|P\| < \delta$. Then $L$ is Riemann's integral of $f$ over

$$\int_a^b f(x)dx = L. \qquad (1.1)$$

Suppose $\mathcal{H}$ is a separable Hilbert space and $T \in (B)\mathcal{H}$. According to [2], $T$ is a *Hilbert- Schmidt operator* if there exist an operator basis

$$\{e_n\}_n^\infty : \sum_{n=1}^{\infty} \|Te_n\|^2 < \infty. \qquad (1.2)$$

Vectors which have complex components are symbolized by $|a\rangle$ and they can also be obtained by linear combination of a set of basis vectors i.e.

$$|a\rangle = c_1 |x_1\rangle + c_2|x_2\rangle \ldots\ldots\ldots. = (x + a)^n = \sum_j c_j|x_j\rangle \ (1.3)$$

Based on the properties of Hilbert space operators studied above, an alternative approach in the formulation of the Schrödinger equation is of great importance.

## II. RELATED LITERATURE REVIEW

### *2.1 Derivation of Electromagnetic Wave Function*

Electromagnetic wave equation is derived from Maxwell equations, that is, as shown from equations (2.1a) – (2.1d) [5].

$$\vec{\nabla}.\vec{E} = 0 \text{ (Gauss' law of electricity)} \qquad (2.1a)$$

$$\vec{\nabla}.\vec{B} = 0 \text{ (Gauss law of magnetism)} \qquad (2.1b)$$

$$\nabla \text{x} \vec{E} = -\frac{\partial \vec{B}}{\partial t} \text{(Faraday's law induction)} \qquad (2.1c)$$

$$\nabla \text{x} \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \text{ (Ampere's' law)} \ (2.1d)$$

Taking the curl for $\vec{E}$ field propagated along the $x$ direction, by [19] we obtain,

$$\nabla \times \vec{E}(x,t)\hat{j} = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial x} & \frac{\partial y}{\partial x} \\ 0 & E(x,t) & 0 \end{bmatrix} = \frac{\partial \vec{E}}{\partial x}\hat{k} \ (2.2)$$

Taking the curl of Faraday's law and substituting Ampere's law for a charge and current free region, [19] obtained

$$\nabla \times \nabla \times \text{E} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} \qquad (2.3)$$

[19] represented three dimensions wave equation, as shown below

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}. \qquad (2.4)$$

***Remark 2.1.1***

The derived electromagnetic wave equation in this literature will be essential in establishment of the correlation between Hilbert space operators and Electromagnetic wave equation. This then will be later used in the derivation of the Schrödinger equation.

**2.2 The Einstein Theory of Relativity**

Einstein relativistic expressions can be derived starting from the relativity principle and the classical Lorentz's law (Hamdan *et al.,* 2007) as shown below

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \qquad (2.5)$$

where,

$q$-charged particle

$\vec{v}$-velocity of the particle

$\vec{E}$-electric field and

$\vec{B}$-magnetic field flux density.

The Cartesian components of equation are given by

$$F_x = q(E_x + v_y, B_z - v_z, B_y) \qquad (a)$$

(2.6)

$$F_y = q(E_y + v_z, B_x - v_x, B_z) \qquad (b)$$

$$F_z = q(E_z + v_x, B_y - v_y, B_x) \qquad (c)$$

Applying relativity principles on equations (2.6) we obtain

$$F'_x = q(E'_x + v'_y, B'_z - v'_z, B'_y) \qquad (a)$$

(2.7)

$$F'_y = q(E'_y + v'_z, B'_x - v'_x, B'_z) \qquad (b)$$

$$F'_z = q(E'_z + v'_x, B'_y - v'_y, B'_x) \qquad (c)$$

According to [8], the relativity principles are represented by equations (2.8a)-(2.8c).where scalar factor $\gamma$ is fixed by applying the relativity principle

$$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$v'_x = \frac{v_x - u}{1 - v_x\frac{u}{c^2}} \qquad (a)$$

(2.8)

$$v'_y = \frac{v_y}{\gamma\left(1-\frac{uv_x}{c^2}\right)} \qquad (b)$$

$$v'_z = \frac{v_z}{\gamma\left(1-\frac{uv_x}{c^2}\right)} \qquad (c)$$

In classical physics, a particle with rest mass $m_0$ with velocity $v$ has a momentum of $p = m_0 v$ and a kinetic energy of $T = \frac{1}{2}m_0 v^2$ and in relativistic physics,

$$p = \frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}} = mv$$

$$p^2 = \gamma^2(m^2{}_0 u^2) = m^2 v^2 \qquad (2.8)$$

The root for the first term presented is

$$\varepsilon = mc^2\sqrt{1-\frac{v'^2}{c^2}} = \gamma m_0 c^2 = mc^2 \qquad (2.9)$$

Equation (2.9) is the relativistic energy $E$, telling us that the change of mass of a particle is accompanied by change in its energy and vice versa.

Using the above classical Cartesian components of Lorentz's law, relativistic velocities, classical momentum and kinetic energy show above, Hamdan *et al*., (2007) derived relativistic energy as shown below.

$$\varepsilon^2 = c^2 p^2 + m^2{}_0 c^4 \qquad (2.10)$$

***Remark 2.2.1***

This relativistic energy derived from the literature will be useful in the derivation of Schrödinger equation using Hilbert space operators.

### 2.3 Derivation of Schrödinger Equation

[20] dealt with the derivation of Schrödinger equation. In their work they used electromagnetic wave equation and Einstein's theory of relativity knowledge. They applied the same approach as that used by Schrödinger. However, [20] extended the wave equation for classical fields to photons and generalized to non-zero rest mass particles and using approximations consistent with non-relativistic particles. [20] considered the one dimension equation**:**

$$\frac{\partial^2 E}{\partial^2 x} = \frac{1}{c^2}\frac{\partial^2 E}{\partial^2 t} \qquad (2.11)$$

This satisfies,

$$E(t,x) = E_0 e^{i(kx-\omega t)} \qquad (2.12)$$

where $k = \frac{2\pi}{\lambda}$ and $\omega = 2\pi\upsilon$ are spatial and temporal frequancies respectively. Substituting equation (2.11) in (2.12) he obtained

$$(\frac{\partial^2}{\partial^2 x} - \frac{1}{c^2}\frac{\partial^2}{\partial^2 t})E_0 e^{i(kx-\omega t)} = 0 \qquad (2.13)$$

On solving the wave vector, the dispersion relation for light in free space is $k = \frac{\omega}{c}$ where $c$ is a wave propagation speed. In this case speed of light is in vacuum. From Einstein and Compton, the energy of photon is $\varepsilon = h\upsilon = \hbar\omega$ and the momentum of photon is

$$p = \frac{h}{\lambda} = \hbar k. \qquad (2.14)$$

Therefore equation (2.12) becomes

$$E(x,t) = E_0 e^{\frac{i}{\hbar}(px-\mathcal{E}t)} \qquad (2.15)$$

And on substituting equation (2.15) in equation (2.13) [8] obtained

$$-\frac{1}{\hbar^2}\left(p^2 + \frac{\omega^2}{c^2}\right) E_0 e^{\frac{i}{\hbar}(px-\varepsilon t)} = 0 \qquad (2.16)$$

where, $\varepsilon^2 = p^2c^2$.

Since [20] were dealing with electric field, they replaced $E$ with $\Psi$, the wave function. Therefore,

$$-\frac{1}{\hbar^2}\left(p^2 + \frac{\omega^2}{c^2}\right) \Psi_0 e^{\frac{i}{\hbar}(px-\varepsilon t)} = 0$$

(2.17)

For relativistic total energy, $\varepsilon^2 = p^2c^2 + m^2c^4$ i.e.

$$\varepsilon = mc^2\sqrt{1 + \frac{p^2}{m^2c^2}} \qquad (2.18)$$

Expanding binomially, we get

$$\simeq mc^2 + \frac{p^2}{m^2c^2} = mc^2 + \text{Ţ} \qquad (2.19)$$

where Ţ is the classical kinetic energy.

Thus equation $\Psi(x,t) = \Psi_0 e^{\frac{i}{\hbar}(px-\varepsilon t)}$ becomes

$$\Psi(x,t) = \Psi_0 = e^{\frac{i}{\hbar}(px-mc^2t-\text{Ţ}t)}$$

$$= e^{-\frac{i}{\hbar}mc^2t}\Psi_0 e^{\frac{i}{\hbar}(px-\text{Ţ}t)} \qquad (2.20)$$

Now if we let $\Psi_0 e^{\frac{i}{\hbar}(px-\text{Ţ}t)} = \Phi$, then

$$\Psi(x,t) = e^{\frac{i}{\hbar}(mc^2t)}\Phi \qquad (2.21)$$

Carrying out the second derivative with respect to $t$ on equation (2.21) Ward & Volmer (2006) obtained

$$\frac{\partial^2 y}{\partial x^2} = \left(-\frac{m^2c^4}{\hbar^2}e^{-\frac{i}{\hbar}mc^2t}\Phi - \frac{2i}{\hbar}mc^2 e^{-\frac{i}{\hbar}mc^2t}\frac{\partial\Phi}{\partial t}\right) + e^{-\frac{i}{\hbar}mc^2t}\Phi$$

(2.22)

The first term in brackets is large and the last term is small. Keeping the large term and discarding the small one, using this approximation in the Klein-Gordon equation

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi - \nabla^2\Psi + \frac{m^2c^4}{\hbar^2}\Psi = 0$$

(2.23)

[20] arrived at the Schrödinger equation for free particle

$$\frac{-\hbar^2}{2m}\nabla^2\Phi = i\hbar\frac{\partial\Phi}{\partial t} \qquad (2.24)$$

where $\Phi$ is a non-relativistic wave function.

***Remark 2.3.1***

The above derivation of Schrödinger equation using electromagnetic wave equation and Einstein theory of relativity, helped in the derivation Schrödinger equation using Hilbert space operators.

### 2.4 Application of Hilbert Spaces Operators

In this section, we present several results based on properties of operators that shall be essential in the sequel especially in obtaining our main objective:

**Theorem 2.4.1:** [13].

Let $S, T \in B(\mathcal{H}), c \in \mathbb{C}$. Then:

i. $T^* \in B(H)$

ii. $(S+T)^* = S^* + T^*, (cT)^* = \bar{c}T^*$

iii. $(ST)^* = T^*S^*$

iv. $T^{**} = T$

v. If $T$ is invertible then $T^*$ is also invertible and $(T^*)^{-1} = (T^{-1})^*$

vi. $\|T\| = \|T^*\|, \|TT^*\| = \|T^*T\| = \|T\|^2$. That is, the $(C^*)$ properties.

**Theorem 2.4.2:** [13].

Let $U \in B(H)$ . The following statements are equivalent:

i. $U$ is unitary

ii. $U$ is bijective and $\langle Ux, Uy\rangle = \langle x, y\rangle$ for every $x, y \in \mathcal{H}$

iii. $U$ surjective and isometric $\|Ux\| = \|x\|$.

**Theorem 2.4.3:** [13].

Let $P \in B(H)$. Then the following are equivalent:

a) $P$ is a projection

b) $I - P$ is a projection

c) $P^2 = P$ and is self adjoint

d) $P^2 = P$ and $P$ is normal

**Proposition 2.4.4:** [9].

(The polarization identity). Let $S$ be a sesquiliear form and let $q(x) = S(x, x)$ then $S(x, y) = \frac{1}{4}[q(x+y) - q(x-y) + iq(x-iy) - iq(x+iy)]$.

**Theorem 2.4.5: (Projection).** [13].

Let $M$ be a closed linear subspace of Hilbert space $\mathcal{H}$. Then every $a \in \mathcal{H}$ can be uniquely written as $a = a_{\parallel} + a_{\perp}$ with $a_{\parallel} \in M$ and $a_{\perp} \in M^{\perp}$ and $\mathcal{H} = M \oplus M^{\perp}$ where $M^{\perp}$ is the orthogonal complement of $M$.

**Theorem 2.4.6:** [16].

Let $M$ be a closed subspace of $\mathcal{H}$. Let $\{e_i : i \in I\}$ be any orthonormal bais for $M$ and let $\{e_j : j \in J\}$be any orthonormal set such that $\{e_i : I \cup J\}$ is orthonormal basis for $\mathcal{H}$. Then the index $I$ and $J$ are disjoint then the following conditions on vector $x \in \mathcal{H}$ are equivalent

$$x \perp y \quad \forall y \in M$$

$$x = \sum_{j \in J} x, \langle x, e_j\rangle e_j.$$

**Theorem 2.4.7:** [6].

(Parallelogram identity). Let $E$ be a normed space. Then there is an inner product on $E$ which gives rise to the norm iff the parallelogram identity $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ is satisfied for all $x, y \in E$.

**Theorem 2.4.8:** [13].

The product of two normal operators is itself normal if and only if the operators commute.

**Theorem 2.4.9:** [6].

If $T$ is self adjoint operator on Hilbert space $\mathcal{H}$ then $\|T\| = sup|\langle Tx, x\rangle| : \|x\| = 1$

**Remark 2.4.10**

For the sake of further reference, the proofs for theorems (2.4.11), (2.4.12) (2.4.13) are provided.

**Theorem 2.4.11:** [13].

If $T$ is idempotent self adjoint operator then $T$ is a projection of $M = x \in \mathcal{H} : Tx = x$

**Proof**

Let $Z \in \mathcal{H}$ and write it as $Z = TZ + (Z - TZ)$

$T(TZ = TZ)$ so $TZ \in M$ and $Z - TZ \in M^{\perp}$.

If $x \in M$, then $\langle x, Z - TZ\rangle = \langle x, Z\rangle - \langle x, TZ\rangle = \langle x, Z\rangle - \langle Tx, Z\rangle = 0$. ∎

**Theorem 2.4.12:** [1].

If $P$ is a nonzero orthogonal projection, then $\|P\| = 1$

**Proof**

If $x \in \mathcal{H}$ and $Px \neq 0$, then the use of the Cauchy-Schwarz inequality implies that

$$\|Px\| = \frac{\langle Px, Px\rangle}{\|Px\|} = \frac{\langle x, P^2x\rangle}{\|Px\|} = \frac{\langle x, Px\rangle}{\|Px\|} \le \|x\|$$

$\|Px\| \le 1$ . If $P \neq 0$ , then there is an $x \in \mathcal{H}$ with $Px \neq 0$ and $\|P(Px)\| = \|Px\|$ thus $\|P\| \ge 1$.

■

**Proposition 2.4.13:** [4]

Suppose that $T$ is a bounded linear operator on a separable Hilbert space $\mathcal{H}$ such that there is an orthonormal $\{e_n\}_{n=1}^{\infty} : \sum_{n=1}^{\infty}\|Te_n\|^2 < \infty$ for any orthogonormal basis $\{f_n\}_{n=1}^{\infty}$ sch that $\sum_{n=1}^{\infty}\|Tf_n\|^2 = \sum_{n=1}^{\infty}\|Te_n\|^2$.

**Theorem 2.4.14:** [3].

(Fixed Point Theorem)

Let $(X, d)$ be a complete metric space and $f: X \to X$ be a map such that $d\big(f(x), f(x_,)\big) \le cd(, x_,)$ for some $0 < c < 1$ and for all $x, x_, \in X$. Then f has a unique fixed point in X. Moreover, for any $x_0 \in X$ . The sequence iterates $x_0, f(x_0), f\big(f(x_0)\big), , , , ,$ converges to the fixed point of $f(x)$.

Where $d(f(x), f(x_,)) \le cd(x, x_,)$, then $f(x)$ is called contraction.

**Theorem 2.4.15:** [12].

If $B$ is any bounded operator and if $A$ is normal and not necessarily bounded and if $BA \subset AB$ then $BA^* \subset A^*B$.

**Proof**

This proof follows from disjoint the Borel sets of complex plane given as

$$Q = K(\propto_1)BK(\propto_2) = 0$$

$K(\propto_1)$ denotes the projection operator with Borel set $\propto$ by spectral family $K_z$.

Suppose $\propto_1$ and $\propto_2$ are bounded then

$$B\int_{\propto_2} ZdK_z x = ABK(\propto_2)x$$

Applying the operator $K(\propto_1)$, we obtain

$$K(\propto_1)B\int_{\propto_2} ZdK_z = \int_{\propto_1} ZdK_z BK \propto_2 .$$

If $z_1$ and $z_2$ are arbitrary numbers in $\propto_1$ and $\propto_2$ respectively, then the above equation can be written as

$$\int_{\propto_1}(Z - Z_1)dK_z Q = Q\int_{\propto_2}(Z - Z_2)dK_z + (Z_2 - Z_1)Q$$

Let $\propto$ denote any Borel set then

$$K(\propto)B = K(\propto)BI = K(\propto)B\big(K(\propto)\big) + K(\propto')$$
$$= K(\propto)BK(\propto)$$

$\propto'$is the compement of $\propto$. Similary,

$$BK(\propto) = K(\propto)B\big(K(\propto)\big)$$

therefore, $K(\propto)B = BK(\propto)$.

This implies that $BA^* \subset A^*B$ ■

**Remark 2.4.16**

The above properties and theorems of Hilbert space shall be used in the derivation and study of applications of the Schrödinger equation using Hilbert space approach.

## III. MAIN RESULTS: THE SCHRODINGER EQUATION AND HILBERT SPACE OPERATORS

### *3.1. Electromagnetic Wave Theory and Einstein Theory of Relativity in Correlation with Operators in Hilbert Space*

This section deals with the correlation of Hilbert space operators with electromagnetic wave equation. The solutions are obtained from the properties of Hilbert space operator as well as the existing derivation of electromagnetic wave equation from the existing literature in section 2.1.

We have also described the correlation of Einstein theory of relativity with Hilbert space operators using the existing literature in section 2.2 and properties of Hilbert space operators.

#### *3.1.1. Correlation between Hilbert Space Operators and Electromagnetic Wave Function Theory*

We recall that a Hilbert Space is a complete inner product space. Dirac invented an alternative for inner product that leads to bras $\langle .|$and kets$|.\rangle$ [14]. That is,

$$\langle x, y\rangle \rightarrow \langle x|y\rangle$$

Bra-kets have the following properties

i. $\langle x|y\rangle = 0$ if both $x$ and $y$ are orthogonal
ii. $\langle x|x\rangle = 0$ iff $x = 0$ (null property)
iii. $\langle x|x\rangle \geq 0 = \|x\|^2$
iv. $\langle x|ay + bz\rangle = a\langle x|y\rangle + b\langle x|z\rangle$.

Properties of dot product are similar to that of inner product. They include:

i. $x.x = |x|^2$
ii. $x.y = y.x$
iii. $a.(b + c) = a.b + a.c$
iv. $ea.b = e(a.b) = a(eb)$ for e is a scalar and $a, b, c$ are vectors.

Electromagnetic waves are electric and magnetic waves that travel perpendicular to each other. By [21], these waves are orthogonal and can be represented as ($\langle E, B\rangle = 0$). They have Amplitude, Wavelength and Frequency.

Electromagnetic wave equation is a second order partial differential equation which describes electromagnetic waves through a medium or a vacuum. The vector differential operator is given as

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k} \quad (3.1)$$

Maxwell equations describe the world of electromagnetic, that is, how electric and magnetic field interact. Applying the properties of Hilbert space on Maxwell equations, they can be represented as follows;

$\langle\vec{\nabla}.\vec{E}\rangle = 0$ (Gauss' law of electricity) (3.2a)

$\langle\vec{\nabla}.\vec{B}\rangle = 0$ (Gauss law of magnetism (3.2b)

$\nabla \mathrm{x}\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ (Faraday's law induction) (3.2c)

$\nabla \mathrm{x}\vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$ (Ampere's law) (3.2d)

In this derivation, we used the procedure used by [19] but applying properties of Hilbert space operators.

For non-conducting media, or in a vacuum, there are no sources and hence,

$$\rho = 0, and\ \sigma = 0$$

Where $\mu\ and\ \varepsilon$ are permeability and permittivity of free space respectively.

Since $\vec{\nabla}$ and $\vec{E}$ are both vectors, the Maxwell equation (3.2a) can be written as;

$$\langle\vec{\nabla}, \vec{E}\rangle = 0 \quad (3.3)$$

Taking the curl of Faraday's law (equation 3.2c) becomes,

$$\vec{\nabla} \times (\vec{\nabla} \times \boldsymbol{\vec{E}}) = -\frac{\partial(\vec{\nabla} \times \boldsymbol{\vec{B}})}{\partial t} \quad (3.4)$$

Considering the left hand side of equation (3.4) we have

$$\vec{\nabla} \times (\vec{\nabla} \times \boldsymbol{\vec{E}}) = \langle\vec{\nabla}, \langle\vec{\nabla}, \vec{E}\rangle\rangle - \langle\vec{E}, \langle\vec{\nabla}, \vec{\nabla}\rangle\rangle \quad (3.5)$$

From property (iii) of inner product equation (3.5) becomes

$$\vec{\nabla} \times (\vec{\nabla} \times \boldsymbol{\vec{E}}) = \langle\vec{\nabla}, \langle\vec{\nabla}, \vec{E}\rangle\rangle - \langle\langle\nabla, \nabla\rangle, \vec{E}\rangle \quad (3.6)$$

By, the first of Maxwell equation, equation (3.2a) tells us that $\langle\nabla, \vec{E}\rangle$= 0 in vacuum. Therefore,

$$\vec{\nabla} \times (\vec{\nabla} \times \boldsymbol{\vec{E}}) = -\langle\nabla^2, \vec{E}\rangle \quad (3.7)$$

Considering the right hand side of equation (3.4), $\frac{\partial(\vec{\nabla} \times \boldsymbol{\vec{B}})}{\partial t}$, substituting the Ampere's law for a charge and current-free region we have

$$\frac{\partial(\vec{\nabla} \times \boldsymbol{\vec{B}})}{\partial t} = \frac{\partial}{\partial t}\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} = \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2} \quad (3.8)$$

Hence using equations (3.4) and (3.8) we obtain,

$$\langle\langle\boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle, \boldsymbol{\vec{E}}\rangle = -\frac{1}{\langle c, c\rangle}\frac{\partial^2\boldsymbol{\vec{E}}}{\partial t^2} \quad (3.9)$$

We find that each component of the electric field satisfies equation (3.9) which is the derived wave equation using properties of inner product. The quantity c is defined as the speed of the wave and $\mu_0\varepsilon_0 = \frac{1}{\langle c, c\rangle}$ or $c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$.

#### 3.1.1. Correlation between Hilbert Space Operators and Einstein Theory of Relativity

Einstein relativistic expressions can be derived starting from the relativity principle and the classical Lorentz's law (Hamdan *et al*., 2007) as shown in equation (3.10)

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \quad (3.10)$$

where,

$q$- charged particle

$\vec{v}$ - velocity of the particle

$\vec{E}$ - electric field and

$\vec{B}$ - magnetic field flux density.

In this work, Einstein theory of relativity is derived following the procedure by Hamdan *et al*., (2007) but using properties of Hilbert space operator.

Since $q$ is a scalar quantity and $\vec{F}, \vec{B}$ and $\vec{v}$ are vectors quantities, applying property, the Cartesian components of equation (3.10) are given by

$$F_x = qE_x + q\langle v_y, B_z\rangle - q\langle v_z, B_y\rangle \quad (3.11a)$$

$$F_y = qE_y + q\langle v_z, B_x\rangle - q\langle v_x, B_z\rangle \quad (3.11b)$$

$$F_z = qE_z + \langle v_x, B_y\rangle - \langle v_y, B_x\rangle \quad (3.11c)$$

Applying relativity principles on equations (3.11a), (3.11b) and (3.11c), we obtain

$$F'_x = qE'_x + \langle v'_y, B'_z\rangle - \langle v'_z, B'_y\rangle \quad (3.12a)$$

$$F'_y = qE'_y + \langle v'_z, B'_x\rangle - \langle v'_x, B'_z\rangle \quad (3.12b)$$

$$F'_z = qE'_z + \langle v'_x, B'_y\rangle - \langle v'_y, B'_x\rangle \quad (3.12c)$$

In the derivation of relativistic energy, the 3-vector relativistic velocity transformation is necessary. According to [8], the relativistic velocity equations but applying the properties of inner product can be written as

$$v'_x = \frac{v_x - u}{1 - \frac{\langle v_x, u\rangle}{\langle c,c\rangle}} \quad (3.13a)$$

$$v'_y = \frac{v_y}{\gamma\left(1 - \frac{\langle u, v_x\rangle}{\langle c,c\rangle}\right)} \quad (3.13b)$$

$$v'_z = \frac{v_z}{\gamma\left(1 - \frac{\langle u, v_x\rangle}{\langle c,c\rangle}\right)} \quad (3.13c)$$

where scalar factor $\gamma$ is fixed by applying the relativity principle $\gamma = \frac{1}{\sqrt{1 - \frac{\langle u,u\rangle}{\langle c,c\rangle}}}$.

In classical physics, a particle with rest mass $m_0$ with velocity $v$ has a momentum of $p = m_0 v$ and a kinetic energy of $T = \frac{1}{2} m_0 v^2$ and in relativistic physics,

$$p = \frac{m_0 v}{\sqrt{1 - \frac{\langle v,v\rangle}{\langle c,c\rangle}}} = \gamma m_0 v = mv \quad (3.14)$$

Lets consider two inertial systems $S$ and $S'$. The charged particle $q$ when viewed from $S$ the components of momentum are given by the following as stated by [18].

$$p_x = mv_x \quad (3.15a)$$

$$p_y = mv_y \quad (3.15b)$$

$$p_z = mv_z \quad (3.15c)$$

When viewed from $S'$, the momentum is given by

$$p'_x = m'v'_x \quad (3.16a)$$

$$p'_y = m'v'_y \quad (3.16b)$$

$$p'_z = m'v'_z \quad (3.16c)$$

From 3.15(a), we have

$$v_x = \frac{p_x}{m} \quad (3.17)$$

While from 3.16(a)

$$v'_x = \frac{p'_x}{m'} \quad (3.18)$$

Equating (3.18) and equation 3.13(a), we obtain

$$\frac{p'_x}{m'} = \frac{v_x - u}{1 - \frac{\langle u, v_x\rangle}{\langle c,c\rangle}} \quad (3.19)$$

Substituting equation (3.17) in (3.19), we obtain

$$\frac{p'_x}{m'} = \frac{p_x - mu}{m\left(1 - \frac{\langle u, v_x\rangle}{\langle c,c\rangle}\right)} \quad (3.20)$$

Observers of frame $S$ measures the rest mass $m_0$, observers from $S'$ measure the mass $m'$. Assuming the charged particle is at rest then

$$v_x = u = 0.$$

Observers of frame $S'$ measures the rest mass $m_0$, observers from $S$ measure the mass $m$. Assuming the charged particle is at rest then the component of momentum if combine 3.13(b), 3.15(b) and 3.16(b), we deduce

$$p'_y = m'v'_y = mv_y = p_y \quad (2.21)$$

In similar way, we get,

$$p'_z = m'v'_z = mv_z = p_z \quad (3.22)$$

The relativistic mass in both frames is expressed as

$$m = \frac{m_0}{\sqrt{1-\frac{\langle v,v\rangle}{\langle c,c\rangle}}} \quad (3.23)$$

$$m' = \frac{m_0}{\sqrt{1-\frac{\langle v,v\rangle}{\langle c,c\rangle}}} \quad (3.24)$$

Multiplying the equation for scalar factor $\gamma$ by $m^2{}_0\langle c,c\rangle\langle c,c\rangle$, we obtain

$$\gamma\sqrt{1-\frac{u^2}{c^2}}(m^2{}_0\langle c,c\rangle\langle c,c\rangle) = m^2{}_0\langle c,c\rangle\langle c,c\rangle$$

(3.25)

$$\langle c,c\rangle\langle c,c\rangle\gamma^2 m^2{}_0 - \langle c,c\rangle\gamma^2 m^2{}_0\langle v,v\rangle = m^2{}_0\langle c,c\rangle\langle c,c\rangle$$

$$= \gamma^2 m^2{}_0\langle c,c\rangle\langle c,c\rangle - \gamma^2 m^2{}_0\langle u,u\rangle\langle c,c\rangle = m^2{}_0\langle c,c\rangle\langle c,c\rangle$$

(3.26)

It is noted that

$$\langle p,p\rangle = \gamma^2(m^2{}_0\langle u,u\rangle) = m^2\langle v,v\rangle \quad (3.27)$$

The root for the first term presented is

$$\varepsilon = m\langle c,c\rangle\sqrt{1-\frac{\langle v,v\rangle}{\langle c,c\rangle}} = \gamma m_0\langle c,c\rangle = m\langle c,c\rangle \quad (3.28)$$

Equation (3.28) is the relativistic energy $\varepsilon$, telling us that the change of mass of a particle is accompanied by change in its energy and vice versa.

Therefore,

$$\varepsilon^2 = \langle c,c\rangle\langle p,p\rangle + m^2{}_0\langle c,c\rangle\langle c,c\rangle$$

(3.29) ■

This is the derived equation for relativistic energy. It which that shall be used in the derivation Schrödinger equation using of Hilbert space operators.

### 3.1.1. The Derivation of Schrodinger Equation using Hilbert Space Operators

The results obtained in section 3.1.1 and 3.1.2 are utilized in the derivation of Schrödinger equation using Hilbert space approach.

[20] dealt with the derivation of Schrödinger equation using electromagnetic wave equation and Einstein's theory of relativity knowledge. They further extended the wave equation for classical fields to photons and generalized it to non-zero rest mass particles using approximations consistent with non-relativistic particles. In this research, we use the same approach as used by [20] but applying the properties of Hilbert space operators.

Equation (3.9) obtain from the derivation of electromagnetic wave equation can be written as

$$\langle\langle\nabla,\nabla\rangle,\vec{\boldsymbol{E}}\rangle - \frac{1}{c^2}\frac{\partial^2\vec{\boldsymbol{E}}}{\partial t^2} = 0 \quad (3.30)$$

This satisfies,

$$E(t,x) = E_0 e^{i(kx-\omega t)} \quad (3.31)$$

where $k = \frac{2\pi}{\lambda}$ and $\omega = 2\pi v$ are spatial and temporal frequencies respectively. Substituting equation (3.31) in (3.30) we obtained

$$\left(\langle\langle\nabla_x,\nabla_x\rangle,E_0\rangle - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E_0 e^{i(kx-\omega t)} = 0$$

(3.32)

In a vacuum, the speed of light is given as $c = v\lambda$, a wave propagation speed and $k = \frac{\omega}{c}$. From Einstein and Compton, the energy of photon is $\varepsilon = hv = \hbar\omega$ and the momentum of photon is $p = \frac{h}{\lambda} = \hbar k$.

Therefore equation (3.31) is written as

$$E(x,t) = E_0 e^{\frac{i}{\hbar}(px-\mathcal{E}t)}$$

(3.33)

Substituting equation (3.33) in (3.32) we obtain,

$$\left(\langle\langle\nabla_x,\nabla_x\rangle,E_0\rangle - \frac{1}{\langle c,c\rangle}\frac{\partial^2}{\partial t^2}\right)E_0 e^{\frac{i}{\hbar}(px-\mathcal{E}t)} = 0 \quad (3.34)$$

Differentiating equation 3.34

$$-\frac{1}{\hbar^2}\left(\langle\langle p,p\rangle,E_0\rangle + \varepsilon^2\left\langle\frac{1}{\langle c,c\rangle},E_0\right\rangle\right)e^{\frac{i}{\hbar}(px-\varepsilon t)} = 0$$

(3.35)

Since $E,\Psi \in S$, where $S$ is a vector space, then replacing electric field, $E$ with $\Psi$, the wave function equation (3.31) in term of wave function can be written as,

$$\Psi(x,t) = \Psi_0 e^{\frac{i}{\hbar}(px-\mathcal{E}t)} \quad (3.36)$$

Therefore equation (3.35) becomes

$$-\frac{1}{\hbar^2}\left(\langle p^2, \Psi_0\rangle - \varepsilon^2 \langle \frac{1}{c^2}, \Psi_0\rangle + m^2\langle c^4, \Psi\rangle\right) e^{\frac{i}{\hbar}(px-\varepsilon t)} = 0 \qquad (3.37)$$

Now, the relativistic total energy obtained from the results in section 3.1.2 is given as

$$\varepsilon^2 = \langle p, p\rangle\langle c, c\rangle + m^2\langle c, c\rangle\langle c, c\rangle$$

Therefore,

$$\varepsilon = m\langle c, c\rangle\sqrt{1 + \frac{\langle p,p\rangle}{m^2\langle c,c\rangle}} \qquad (3.38)$$

$$\simeq m\langle c, c\rangle\left(1 + \frac{\langle p, p\rangle}{2m^2\langle c, c\rangle}\right)$$

$$\simeq mc^2 + \frac{\langle p,p\rangle}{2m^2\langle c,c\rangle} = m\langle c, c\rangle + \mathrm{T} \qquad (3.39)$$

where $\mathrm{T}$ is the classical kinetic energy.

Thus equation (3.36) becomes

$$\Psi(x, t) = \Psi_0 e^{\frac{i}{\hbar}(px - mc^2 t - \mathrm{T}t)} \qquad (3.40)$$

$$= e^{-\frac{i}{\hbar}mc^2 t}\Psi_0 e^{\frac{i}{\hbar}(px - \mathrm{T}t)} \qquad (3.41)$$

Taking $\Psi_0 e^{\frac{i}{\hbar}(px-\mathrm{T}t)} = \Phi$ then,

$$\Psi(x, t) = e^{-\frac{i}{\hbar}mc^2 t}\Phi \qquad (3.42)$$

On differentiating equation (3.42) with respect to $t$we obtain,

$$\frac{\partial\Psi}{\partial t} = -\frac{m}{\hbar}\langle\langle c, c\rangle, \Phi\rangle e^{-\frac{i}{\hbar}m\langle c,c\rangle t} + e^{-\frac{i}{\hbar}m\langle c,c\rangle t}\frac{\partial\Phi}{\partial t}$$

(3.43)

Carrying out the second derivative of equation 3.43 we have

$$\frac{\partial^2\Psi}{\partial t^2} = \left(-\frac{m^2}{\hbar^2}e^{-\frac{i}{\hbar}mc^2 t}\langle\langle c, c\rangle\langle c, c\rangle, \Phi\rangle - \frac{2i}{\hbar}e^{-\frac{i}{\hbar}m\langle c,c\rangle t}m\langle c, c\rangle\frac{\partial\Phi}{\partial t}\right) + e^{-\frac{i}{\hbar}m\langle c,c\rangle t}\frac{\partial^2\Phi}{\partial t^2} \qquad (3.44)$$

The term $e^{-\frac{i}{\hbar}m\langle c,c\rangle t}\frac{\partial^2\Phi}{\partial t^2}$ is very small therefore it can be discarded. The term in brackets is very large thus, using this approximation in the Klein-Gordon equation we obtain

$$e^{-\frac{i}{\hbar}mc^2 t}\left[\langle\langle\nabla, \nabla\rangle, \Phi\rangle + \frac{2im}{\hbar}\frac{\partial\Phi}{\partial t}\right] = 0$$

(3.45)

$$\langle\langle\nabla, \nabla\rangle, \Phi\rangle + \frac{2im}{\hbar}\frac{\partial}{\partial t}\Phi = 0 \qquad (3.46)$$

Therefore equation (3.46) is the derived Schrödinger equation for free particle which can be written as

$$\frac{-\hbar^2}{2m}\langle\langle\nabla, \nabla\rangle, \Phi\rangle = i\hbar\frac{\partial}{\partial t}\Phi \qquad (3.47)$$

where $\Phi$ is the non-relativistic wave function.

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