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NORM ATTAINABILITY OF GENERALIZED FINITE OPERATORS ON C*-ALGEBRA

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ABSTRACT

Norm attainability of elementary operators on Hilbert and Banach spaces has been characterized before. However, there is little information on Norm attainability of generalized finite operators on C*-algebra. This paper reports the norm attainability of generalized finite operators on C*-algebra. The approach of Okello 2018 has been used to determine norm attainability. Given two pairs of norm attainable operators A, B, implementing the generalized finite operators$||AX - XB - I|| \geq 1$, it then follows that the generalized finite operator is also norm attainable.

Keywords: Banach spaces, Complex Hilbert Space, Norm attainable

INTRODUCTION

Let H be a complex Hilbert space, $B(H)$ be the collection of bounded linear operators on H with, inner product space and GF be the set of norm attainable generalized finite operators, the inner derivation is defined by $\delta_A \epsilon(X) = ||XA - AX||$, the generalized derivation $\delta_{AB}(X) = ||AX - XB||$, the generalized finite operators $||AX - XB - I|| \geq 1$ is said to be norm attainable, if for every pair of operators $A, B \epsilon B(H)$ implementing the generalized finite operators are norm attainable and there exists a scalar q and some unit sequence Z_n such that $||Z_n||$=1, |q|=1 and $||(A-q) * Z_n|| < \frac{1}{n}$, and $||(B-q)Z_n|| > \frac{1}{n}$ ||

Definition 1.1 (Involution on algebra, Gelfand *et al*, 1943)

If A is an algebra, a mapping $*: A \rightarrow A$, defined by $x \rightarrow x^*$ is called an involution on algebra A if it satisfies the following four conditions; $\forall\, x, y \epsilon A$.

i) $(x+y)^* = x^* + y^*$

ii) $(\lambda x)^* = \lambda x^*$

iii) $(xy)^* = y^* x^*$

iv) $(x^*)^* = x^{**} = x$

If A is a Banach algebra with an involution and, for every $\forall x \epsilon A$ $\| x^* x \| = \| x \|^2$, then A is called $C^*\!-\!algebra$.

Examples of C*-algebra

i) Let $B(H)$ be a collection of bounded linear operators on a complex Hilbert space H, with inner product space, then $B(H)$ is a C*-algebra.

Definition 1.2 Generalized finite operators GF

Given pairs of operators $(A, B) \epsilon B(H) \times B(H)$: $||AX - XB - I|| \geq 1$ is a generalized finite operator

Definition 1.3

An operator $A \epsilon B(H)$ is said to be norm attainable if for every unit vector $x \epsilon H$ it then follows $||Ax|| = ||A||$

RESULTS AND DISCUSSION

Theorem 2.1 (Okelo 2018)

Let $S, T \epsilon B(H)$ if both S and T are norm attainable then the basic elementary operator M_{ST} is also norm attainable. The lemma below gives the result on norm attainability of inner derivative

Lemma 2.2

Let H be a complex Hilbert space, $B(H)$ be the collection of bounded linear operators on H with, inner product space and GF be the set of norm attainable generalized finite operators, the inner derivation $\delta_A \epsilon GF$ is norm attainable if there exists a scalar q and some sequence Z_n such that $||Z_n||=1$, $|q|=1$ and $||(A-q)*Z_n|| < \frac{1}{n,}$ $AX \rightarrow -AX$

Proof.

We define inner derivative δ_A as $\delta_A(X) = ||AX - XA||$, from $||(A-q)*Z_n|| < \frac{1}{n,}$,when n≥1, then we will have,

$$
\begin{aligned}
||AX - XA||^2 &= ||(A-q)^*XZ_n - Z_n||^2 - ||X(A-q)Z_n||^2 \\
&= ||(AX - qX)Z_n - Z_n||^2 - ||(AX - qX)\,Z_n\,||^2 \\
&= ||\,(AX - qX)\,Z_n||^2 + 1 - \{||(AX - qX)Z_n||^2 \\
&= ||(AX - qX)||^2\,||Z_n||^2 + 1 - \{||(AX - qX||^2||Z_n||^2 \\
&= ||AX||^2 - |q|^2||X||^2 + 1 - \{||AX||^2 - |q|^2||X||^2\} \\
&= ||AX - (-AX) + qX||^2 \\
&= ||AX + AX||^2 + 1 \\
&= 4||A||^2 + 1
\end{aligned}
$$

Getting the square root on both sides of the equation, we have

$$||AX - XA|| = 2||A|| + 1 = \delta_A$$

Since operator A is norm attainable, it then follows that the inner derivative δ_A is norm attainable.

Next we give the conditions for norm attainability of generalized derivative δ_{AB}

Lemma 2.3
Let H be a complex Hilbert space, $B(H)$ be the collection of bounded linear operators on H, with inner product space and GF be the set of norm-attainable generalized finite operators, the generalized derivative $\delta_{AB} \epsilon GF$ is norm attainable if there exists some scalar q and a unit sequence Z_n such that $||Z_n||$=1, $|q| = 1$, $||(A-q)* \quad || > \frac{1}{n}$ and Z_n
$||(B-q)\, Z_n|| > \frac{1}{n}$

Proof
We define a generalized derivative δ_{AB} as $\delta_{AB}(X)$=$||AX - XB||$ for every $x \epsilon B(H)$
It then follows that $||AX - XB||^2 = ||(A-q)XZ_n - Z_n||^2 - \{||X(B-q)\,Z_n||^2\}$
$$= ||(AX - qX)\,Z_n||^2 + 1 - \{||(BX - qX)\,Z_n||^2\}$$
$$= ||(AX - qX)||^2\,||Z_n||^2 + 1 - \{||(BX - qX||^2||Z_n||^2$$
$$= ||AX||^2 - |q|^2||X||^2 + 1 - \{||BX||^2 - |q|^2||X||^2\} \qquad \text{(i)}$$
$$= ||AX||^2 - 1 + 1 - ||BX||^2 + 1$$
$$= ||AX||^2 - ||BX||^2 + 1$$
Getting the square roots on both sides of the equation, we obtain
$$||AX - XB|| = ||A|| - ||B|| + 1 \qquad \text{(ii)}$$
From equation 1 we get the inequality
$$||AX - XB||^2 \geq ||AX||^2 - |q|^2||X||^2 + 1 - \{||BX||^2 - |q|^2||X||^2\}$$
Implying that, $||AX - XB|| \geq ||A|| - ||B|| + 1$ (iii)
For the reverse inequality, from equation 1, we have
$$||AX - XB||^2 \leq ||AX||^2 + |q|^2||X||^2 + 1 - \{||BX||^2 + |q|^2||X||^2\}$$
$$\leq ||AX||^2 + 2 - ||BX||^2 - 1$$
$$\leq ||A||^2 - ||B||^2 + 1$$
Getting square root on both sides we get
$$||AX - XB|| \leq ||A|| - ||B|| + 1 \qquad \text{(iv)}$$
From equation (iii) and (iv) we get
$$||AX - XB|| = ||A|| - ||B|| + 1$$
Hence $||AX - XB|| = ||A|| - ||B|| + 1 = \delta_{AB}$. Therefore δ_{AB} is norm attainable since A and B are norm attainable. The next theorem gives main results on norm attainability of generalized finite operators.

Theorem 2.4
Let H be a complex Hilbert space, $B(H)$ be the collection of bounded linear operators on H with inner product space and $A, B \epsilon GF$, if A and B are norm attainable, then the generalized finite operators $(AB) \epsilon B(H) \times B(H)$: $||AX - XB - I|| \geq 1$ is norm attainable.

Proof
For $A, B \epsilon B(H)$, it is known that $||AX - XB|| = ||A|| - ||B|| + 1$
We let $||Z_n||$=1, $|q| = 1$, $||(A-q)*\,Z_n|| > \frac{1}{n}$ and $||(B-q)\,Z_n|| > \frac{1}{n}$
Now for every n≥ 1, then we will have
$$||AX - XB - I|| \geq Sup\,\{||(AX - XB - I)Z_n||\}$$

$$\geq Sup\ \{||(A-q)XZ_n - Z_n|| - ||X(B-q)\,Z_n|| + 1\}$$
$$\geq Sup\ \{||A|| - ||B|| + 1\}$$

Implying that $||AX - XB - I|| \geq ||A|| - ||B|| + 1$ (i)

For the reverse inequality,

$$||AX - XB - I|| \leq Sup\ \{||(A-q)XZ_n - Z_n|| - ||X(B-q)\,Z_n|| + 1\}$$
$$\leq Sup\ \{||AX|| + |q|||X|| - [||BX|| + |q|||X||] + 1\}$$
$$\leq Sup\ \{||A|| - ||B|| + 1 - 1 + 1$$
$$\leq Sup\ \{||A|| - ||B|| + 1\}$$

Implying that, $||AX - XB - I|| \leq ||A|| - ||B|| + 1$ (ii)

From equation (i) and (ii) we get

$$\left\|AX - XB - I\right\| = ||A|| - ||B|| + 1$$

Therefore the generalized finite operator $A, B \epsilon B(H): ||AX - XB - I|| \geq 1$ is norm attainable.

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