# TRANSITIVITY ACTION OF THE CARTESIAN PRODUCT OF THE ALTERNATING GROUP ACTING ON A CARTESIAN PRODUCT OF ORDERED SETS OF TUPLES

***Maraka K. Moses[1], Musundi W. Sammy[1] and Lewis N. Nyaga[2]***
*[1]Department of Physical Sciences, Chuka University, P. O. Box 109-60400, Chuka*
*[2]Department of Pure and Applied Mathematics, Jomo Kenyatta University of Agriculture and Technology, P. O. Box 62000-00200, Nairobi, Kenya*
*Email: marakamose1@gmail.com, swmusundi@chuka.ac.ke, lnyaga@jkuat.ac.ke*

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**ABSTRACT**
Transitivity action properties of the alternating group $A_n$ on ordered and unordered $n-tuples$ and on the direct product of alternating group on unordered sets have been greatly studied by different researchers. However, no work has been done for transitivity action of the Cartesian product of the alternating group on the Cartesian product of ordered $n-tuples$ of sets. This paper determined the transitivity action of the Cartesian product of the alternating group acting on a Cartesian product of ordered sets of triples. The Orbit-Stabilizer Theorem has been used to determine the transitivity action. When $n \geq 5$, the action of the Cartesian product of alternating group on the Cartesian product of ordered sets of triples is transitive.
**Keywords:** Orbits, Stabilizer, Transitivity action, Fixed point

## INTRODUCTION

### Notation and Terminology

In this paper, we shall represent the following notations as: $\sum$- sum over i; $A_n$ -an alternating group of degree $n$ and order $\frac{n!}{2}$; $|G|$ – the order of a group $G$; $|G:H|$ -Index of $H$ in $G$; $P^{[3]}$ – the set of an ordered triple from set $P = \{1,2,3,\ldots,n\}$; $S^{[3]}$ – the set of an ordered triple from set $S = \{n+1, n+2, \ldots, 2n\}$; $V^{[3]}$ – the set of an ordered triple from set $V = \{2n+1, 2n+2, \ldots, 3n\}$; $[a,b,c]$ -Ordered triple; $A_n \times A_n \times A_n$ -Cartesian product of alternating group $A_n$; $P^{[3]} \times S^{[3]} \times V^{[3]}$ -Cartesian product of ordered sets of triples $P^{[3]}$, $S^{[3]}$ and $V^{[3]}$.

**Definition 1.1.1:** Let $P$ be a non-empty set. A group $G$ is said to act on the left of $P$ if for each $g \in G$ and each $p \in P$ there corresponds a unique element $gp \in G$ such that:

*(i)* $(g_1g_2) = g_1(g_2p)$, $g_1, g_2 \in G$ and $p \in P$.
*(ii)* For any $p \in P$, $ep = P$, where $e$ is the identity in $G$.

The action of $G$ from the right on $P$ can be defined in the same manner.

**Definition 1.1.2:** Let $G$ act on a set $P$. Then $P$ is partitioned into disjoint equivalent classes called orbits or transitivity classes of the action. For every $p \in P$ the orbit containing $p$ is called the orbit of $p$ and is denoted by $Orb_G(p)$.

**Definition 1.1.3:** Let $G$ act on a set $P$ and $p \in P$. The stabilizer of $p$ in $G$, denoted by $Stab_G(p)$ is given by $Stab_G(p) = \{g \in G | gp = p\}$.

**Definition 1.1.4:** Let $G$ act on a set $P$. The set of elements of $P$ fixed by $g \in G$ is called the fixed-point set of $G$ and is denoted by . Thus $Fix(g) = \{p \in P | gp = p\}$.

$Fix(g)$

**Definition 1.1.5:** If the action of a group $G$ on set $P$ has only one orbit, then we say that $G$ acts transitively on $P$ In other words, $G$ acts transitively on $P$ if for every pair of points $p, s \in P$, there exists $g \in G$ such that $gp = s$.

**Definition 1.1.6:** A group $G$ with two subgroups $H$ and $K$, then they are said to be conjugate if $H = gkg^{-1}$ for some $g \in G$.

**Theorem 1.1.7 (Krishnamurthy, 1985, p. 68):** Two permutations in $A_n$ are conjugate if and only if, they have the same cycle type and if $g \in S_n$ has cycle type $(\alpha_1, \alpha_2, \dots, \alpha_n)$, then the number of permutations in $S_n$ conjugate to $g$ is, $\frac{n!}{\prod_{i=1}^{n} \alpha_i! i^{\alpha_i}}$.

**Theorem 1.1.8 (Orbit – Stabilizer Theorem, Rose, 1978, p.72):** Let $G$ act on a set $P$. Then $|Orb_G(p)| = |G: Stab_G(p)|$.

**Theorem 1.1.9 (Cauchy-Frobenius Lemma, Rotman, 1973, p.45):** Let $G$ be a group acting on a finite set $P$Then the number of $G$-orbits in $P$ is,

$$\frac{1}{|G|} \sum_{g \in G} |Fix(g)|.$$

**Definition 1.1.10 (Direct product action, Cameron *et al*, 2008):** Let $(G_1, P_1)$ and $(G_2, P_2)$ be permutation groups. The direct product $G_1 \times G_2$ acts on the disjoint union $P_1 \cup P_2$ by the rule $p(g_1, g_2) = \begin{cases} pg_1, if\ p \in P_1, \\ pg_2, if\ p \in P_2 \end{cases}$ and on the Cartesian product $P_1 \times P_2$ by the rule $(p_1, p_2)(g_1, g_2) = (p_1 g_1, p_2 g_2)$.

**Theorem 1.1.11 (Armstrong, 2013):** The $G_1 \times G_2 \times G_3$ -orbit containing $(p, s, v) \in P \times S \times V$ is given by $Orb_{G_1}(p) \times Orb_{G_2}(s) \times Orb_{G_3}(v)$ and the stabilizer of $(p, s, v)$ is given by $Stab_{G_1}(p) \times Stab_{G_2}(s) \times Stab_{G_3}(v)$.

**INTRODUCTION**

Higman (1964) introduced the rank of a group on finite permutation groups of rank $3$. In 1970, Hitman proved that the rank of the symmetric group $S_n$ acting on $2$-element subsets from the set $P = \{1,2,\dots,n\}$ is $3$ and the sub degrees are: , $1\ 2(n-1)$ and $\binom{n-2}{2}$. Cameron (1972) worked on the sub orbits of multiply transitive permutationsand later in 1974 studied the sub orbits of primitive groups.

Ndarinyo *et al.*, (2015) showed that the alternating group $A_n = 5,6,7$ acts transitively on unordered and ordered triples from the set $P = 1,2,\dots,n$ when $n \leq 7$ through determination of the number of orbits. Nyaga (2018) proved that the direct product action of the alternating group on the Cartesian product of three sets is transitive. The ranks and sub degrees associated with this action for $n \geq 4$ is $8$; and $1, (n-1), (n-1)^2, (n-1)^3$ respectively. Based on these results we investigate some properties of $A_n \times A_n \times A_n$, the Cartesian product action of thealternating group acting on $P^{[3]} \times S^{[3]} \times V^{[3]}$, the Cartesian product of ordered sets of triples.

The cartesian product of alternating group $A_n \times A_n \times A_n$, acts on $P^{[3]} \times S^{[3]} \times V^{[3]}$, by the rule;
$g_1\{([1,2,3],[1,2,4],\dots,[n,n-1,n-3],[n,n-1,n-2])\} \times g_2\{([n+1,n+2,n+3],[n+1,n+2,n+4],\dots,[2n,2n-1,2n-3],[2n,2n-1,2n-2])\} \times g_3\{([2n+1,2n+2,2n+3],[2n+1,2n+2,2n+4],\dots,[3n,3n-1,3n-3],[3n,3n-1,3n-3])\} = \{g_1([1,2,3],[1,2,4],\dots,[n,n-1,n-3],[n,n-1,n-2]), g_2([n+1,n+2,n+3],[n+1,n+2,n+4],\dots,[2n,2n-1,2n-3],[2n,2n-1,2n-2]), g_3([2n+1,2n+2,2n+3],[2n+1,2n+2,2n+4],\dots,[3n,3n-1,3n-3],[3n,3n-1,3n-3])\}$ ;

$\forall g_1, g_2, g_3 \in A_n, \{([1,2,3],[1,2,4],\dots,[n,n-1,n-3],[n,n-1,n-2])\} \in P^{[3]}$, set of ordered triples from the set $P = \{1,2,3,\dots,n\}$; $\{([n+1,n+2,n+3],[n+1,n+2,n+4],\dots,[2n,2n-1,2n-3],[2n,2n-1,2n-2])\} \in S^{[3]}$, set of ordered triples from the set $S = \{n+1,n+2,\dots,2n\}$;
and
$\{([2n+1,2n+2,2n+3],[2n+1,2n+2,2n+4],\dots,[3n,3n-1,3n-3],[3n,3n-1,3n-3])\} \in V^{[3]}$, set of ordered triples from the set $V = \{2n+1,2n+2,\dots,3n\}$
.

**MAIN RESULTS**

**Lemma 2.1:** The action of $A_5 \times A_5 \times A_5$ on $P^{[3]} \times S^{[3]} \times V^{[3]}$ is transitive.

**Proof:** Let $G = A_5 \times A_5 \times A_5$ act on $P^{[3]} \times S^{[3]} \times V^{[3]}$ where $P^{[3]} = \{$[ 1, 2, 3 ], [ 1, 2, 4 ], [ 1, 2, 5 ], [ 1, 3, 2 ], [ 1, 3, 4 ], [ 1, 3, 5 ], [ 1, 4, 2 ], [ 1, 4, 3 ], [ 1, 4, 5 ], [ 1, 5, 2 ], [ 1, 5, 3 ], [ 1, 5, 4 ], [ 2, 1, 3 ], [ 2, 1, 4 ], [ 2, 1, 5 ], [ 2, 3, 1 ], [ 2, 3, 4 ], [ 2, 3, 5 ], [ 2, 4, 1 ], [ 2, 4, 3 ], [ 2, 4, 5 ], [ 2, 5, 1 ], [ 2, 5, 3 ], [ 2, 5, 4 ], [ 3, 1, 2 ], [ 3, 1, 4 ], [ 3, 1, 5 ], [ 3, 2, 1 ], [ 3, 2, 4 ], [ 3, 2, 5 ], [ 3, 4, 1 ], [ 3, 4, 2 ], [ 3, 4, 5 ], [ 3, 5, 1 ], [ 3, 5, 2 ], [ 3, 5, 4 ], [ 4, 1, 2 ], [ 4, 1, 3 ], [ 4, 1, 5 ], [ 4, 2, 1 ], [ 4, 2, 3 ], [ 4, 2, 5 ], [ 4, 3, 1 ], [ 4, 3, 2 ], [ 4, 3, 5 ], [ 4, 5, 1 ], [ 4, 5, 2 ], [ 4, 5, 3 ], [ 5, 1, 2 ], [ 5, 1, 3 ], [ 5, 1, 4 ], [ 5, 2, 1 ], [ 5, 2, 3 ], [ 5, 2, 4 ], [ 5, 3, 1 ], [ 5, 3, 2 ], [ 5, 3, 4 ], [ 5, 4, 1 ], [ 5, 4, 2 ], [ 5, 4, 3 ]$\}$ ;

$S^{[3]}$ = { [ 6, 7, 8 ], [ 6, 7, 9 ], [ 6, 7, 10 ], [ 6, 8, 7 ], [ 6, 8, 9 ], [ 6, 8, 10 ], [ 6, 9, 7 ], [ 6, 9, 8 ], [ 6, 9, 10 ], [ 6, 10, 7 ],[ 6, 10, 8 ], [ 6, 10, 9 ], [ 7, 6, 8 ], [ 7, 6, 9 ], [ 7, 6, 10 ], [ 7, 8, 6 ], [ 7, 8, 9 ], [ 7, 8, 10 ], [ 7, 9, 6 ], [ 7, 9, 8 ], [ 7, 9, 10 ], [ 7, 10, 6 ], [ 7, 10, 8 ],[ 7, 10, 9 ], [ 8, 6, 7 ], [ 8, 6, 9 ], [ 8, 6, 10 ], [ 8, 7, 6 ], [ 8, 7, 9 ], [ 8, 7, 10 ], [ 8, 9, 6 ], [ 8, 9, 7 ], [ 8, 9, 10 ],[ 8, 10, 6 ], [ 8, 10, 7 ], [ 8, 10, 9 ], [ 9, 6, 7 ], [ 9, 6, 8 ], [ 9, 6, 10 ], [ 9, 7, 6 ], [ 9, 7, 8 ], [ 9, 7, 10 ], [ 9, 8, 6 ], [ 9, 8, 7 ], [ 9, 8, 10 ], [ 9, 10, 6 ],[ 9, 10, 7 ], [ 9, 10, 8 ], [ 10, 6, 7 ], [ 10, 6, 8 ], [ 10, 6, 9 ], [ 10, 7, 6 ], [ 10, 7, 8 ], [ 10, 7, 9 ], [ 10, 8, 6 ], [ 10, 8, 7 ],[ 10, 8, 9 ], [ 10, 9, 6 ], [ 10, 9, 7 ], [ 10, 9, 8 ]} ; and;

$V^{[3]}$ = {[ 11, 12, 13 ],[ 11, 12, 14 ],[ 11, 12, 15 ], [ 11, 13, 12 ], [ 11, 13, 14 ],[11, 13, 15],[ 11, 14, 12 ], [ 11, 14, 13 ], [ 11, 14, 15 ], [ 11, 15, 12 ],[ 11, 15, 13 ], [ 11, 15, 14 ], [ 12, 11, 13 ], [ 12, 11, 14 ], [ 12, 11, 15 ], [ 12, 13, 11 ], [ 12, 13, 14 ], [ 12, 13, 15 ], [ 12, 14, 11 ], [ 12, 14, 13 ], [ 12, 14, 15 ],[ 12, 15, 11 ], [ 12, 15, 13 ],[ 12, 15, 14 ], [ 13, 11, 12 ], [ 13, 11, 14 ], [ 13, 11, 15 ], [ 13, 12, 11 ], [ 13, 12, 14 ], [ 13, 12, 15 ], [ 13, 14, 11 ],[ 13, 14, 12 ], [ 13, 14, 15 ], [ 13, 15, 11 ], [ 13, 15, 12 ], [ 13, 15, 14 ], [ 14, 11, 12 ], [ 14, 11, 13 ], [ 14, 11, 15 ], [ 14, 12, 11 ], [ 14, 12, 13 ], [ 14, 12, 15 ],[ 14, 13, 11 ], [ 14, 13, 12 ], [ 14, 13, 15 ],[ 14, 15, 11 ], [ 14, 15, 12 ], [ 14, 15, 13 ], [ 15, 11, 12 ], [ 15, 11, 13 ], [ 15, 11, 14 ], [ 15, 12, 11 ],[ 15, 12, 13 ], [ 15, 12, 14 ], [ 15, 13, 11 ], [ 15, 13, 12 ], [ 15, 13, 14 ], [ 15, 14, 11 ], [ 15, 14, 12 ], [ 15, 14, 13]} .

The Cartesian product of $P^{[3]} \times S^{[3]} \times V^{[3]}$ is generated using the GAP software with $|P^{[3]} \times S^{[3]} \times V^{[3]}| = 216\,000$. $G$ is generated by $< \{(1\ 2\ 3\ 4\ 5), (123)\}, \{(6\ 7\ 8\ 9\ 10), (678)\}, \{(11\ 12\ 13\ 14\ 15), (11\ 12\ 13)\} >$ using the GAP software. $([1,2,3], [6,7,8], [11,12,13])$ is fixed by an element $(g_p, g_s, g_v) \in G$ if and only if $1,2$ and $3$ comes from a single cycle in $g_p$ ; $6,7$ and $8$ comes from a single cycle in $g_s$ and $11,12$ and $13$ comes from a single cycle in $g_v$ .

Therefore, $Stab_G([1,2,3], [6,7,8], [11,12,13]) = \{1,6,11\} = \{(e_p, e_s, e_v)\}$.
$|Stab_G([1,2,3], [6,7,8], [11,12,13])| = 1$ .

By Orbit-Stabilizer Theorem,
$$|Orb_G([1,2,3], [6,7,8], [11,12,13])| = |G: Stab_G([1,2,3], [6,7,8], [11,12,13])|$$
$$= \frac{|G|}{|Stab_G([1,2,3], [6,7,8], [11,12,13])|}$$
$$= \frac{216000}{1} = 216000 = |P^{[3]} \times S^{[3]} \times V^{[3]}|$$
Therefore, $A_5 \times A_5 \times A_5$ acts transitively on $P^{[3]} \times S^{[3]} \times V^{[3]}$ .

**Lemma 2.2:** The action of $A_6 \times A_6 \times A_6$ on $P^{[3]} \times S^{[3]} \times V^{[3]}$ is transitive.
**Proof:** Let $G = A_6 \times A_6 \times A_6$ act on $P^{[3]} \times S^{[3]} \times V^{[3]}$ where;
$P^{[3]}$ =[ 1, 2, 3 ], [ 1, 2, 4 ], [ 1, 2, 5 ], [ 1, 2, 6 ], [ 1, 3, 2 ], [ 1, 3, 4 ], [ 1, 3, 5 ], [ 1, 3, 6 ],
[ 1, 4, 2 ], [ 1, 4, 3 ], [ 1, 4, 5 ], [ 1, 4, 6 ], [ 1, 5, 2 ], [ 1, 5, 3 ], [ 1, 5, 4 ], [ 1, 5, 6 ],
[ 1, 6, 2 ], [ 1, 6, 3 ], [ 1, 6, 4 ], [ 1, 6, 5 ], [ 2, 1, 3 ], [ 2, 1, 4 ], [ 2, 1, 5 ], [ 2, 1, 6 ],
[ 2, 3, 1 ], [ 2, 3, 4 ], [ 2, 3, 5 ], [ 2, 3, 6 ], [ 2, 4, 1 ], [ 2, 4, 3 ], [ 2, 4, 5 ], [ 2, 4, 6 ],
[ 2, 5, 1 ], [ 2, 5, 3 ], [ 2, 5, 4 ], [ 2, 5, 6 ], [ 2, 6, 1 ], [ 2, 6, 3 ], [ 2, 6, 4 ], [ 2, 6, 5 ],
[ 3, 1, 2 ], [ 3, 1, 4 ], [ 3, 1, 5 ], [ 3, 1, 6 ], [ 3, 2, 1 ], [ 3, 2, 4 ], [ 3, 2, 5 ], [ 3, 2, 6 ],
[ 3, 4, 1 ], [ 3, 4, 2 ], [ 3, 4, 5 ], [ 3, 4, 6 ], [ 3, 5, 1 ], [ 3, 5, 2 ], [ 3, 5, 4 ], [ 3, 5, 6 ],
[ 3, 6, 1 ], [ 3, 6, 2 ], [ 3, 6, 4 ], [ 3, 6, 5 ], [ 4, 1, 2 ], [ 4, 1, 3 ], [ 4, 1, 5 ], [ 4, 1, 6 ],
[ 4, 2, 1 ], [ 4, 2, 3 ], [ 4, 2, 5 ], [ 4, 2, 6 ], [ 4, 3, 1 ], [ 4, 3, 2 ], [ 4, 3, 5 ], [ 4, 3, 6 ],
[ 4, 5, 1 ], [ 4, 5, 2 ], [ 4, 5, 3 ], [ 4, 5, 6 ], [ 4, 6, 1 ], [ 4, 6, 2 ], [ 4, 6, 3 ], [ 4, 6, 5 ],
[ 5, 1, 2 ], [ 5, 1, 3 ], [ 5, 1, 4 ], [ 5, 1, 6 ], [ 5, 2, 1 ], [ 5, 2, 3 ], [ 5, 2, 4 ], [ 5, 2, 6 ],
[ 5, 3, 1 ], [ 5, 3, 2 ], [ 5, 3, 4 ], [ 5, 3, 6 ], [ 5, 4, 1 ], [ 5, 4, 2 ], [ 5, 4, 3 ], [ 5, 4, 6 ],
[ 5, 6, 1 ], [ 5, 6, 2 ], [ 5, 6, 3 ], [ 5, 6, 4 ], [ 6, 1, 2 ], [ 6, 1, 3 ], [ 6, 1, 4 ], [ 6, 1, 5 ],
[ 6, 2, 1 ], [ 6, 2, 3 ], [ 6, 2, 4 ], [ 6, 2, 5 ], [ 6, 3, 1 ], [ 6, 3, 2 ], [ 6, 3, 4 ], [ 6, 3, 5 ],
[ 6, 4, 1 ], [ 6, 4, 2 ], [ 6, 4, 3 ], [ 6, 4, 5 ], [ 6, 5, 1 ], [ 6, 5, 2 ], [ 6, 5, 3 ], [ 6, 5, 4 ]};

$S^{[3]}$ ={[ 7, 8, 9 ], [ 7, 8, 10 ], [ 7, 8, 11 ], [ 7, 8, 12 ], [ 7, 9, 8 ], [ 7, 9, 10 ], [ 7, 9, 11 ], [ 7, 9, 12 ], [ 7, 10, 8 ], [ 7, 10, 9 ],[ 7, 10, 11 ], [ 7, 10, 12 ], [ 7, 11, 8 ], [ 7, 11, 9 ], [ 7, 11, 10 ], [ 7, 11, 12 ], [ 7, 12, 8 ], [ 7, 12, 9 ], [ 7, 12, 10 ], [ 7, 12, 11 ], [ 8, 7, 9 ], [ 8, 7, 10 ], [ 8, 7, 11 ], [ 8, 7, 12 ], [ 8, 9, 7 ], [ 8, 9, 10 ], [ 8, 9, 11 ], [ 8, 9, 12 ], [ 8, 10, 7 ],

[ 8, 10, 9 ], [ 8, 10, 11 ], [ 8, 10, 12 ], [ 8, 11, 7 ],[ 8, 11, 9 ], [ 8, 11, 10 ], [ 8, 11, 12 ], [ 8, 12, 7 ], [ 8, 12, 9 ], [ 8, 12, 10 ], [ 8, 12, 11 ], [ 9, 7, 8 ], [ 9, 7, 10 ], [ 9, 7, 11 ], [ 9, 7, 12 ], [ 9, 8, 7 ], [ 9, 8, 10 ],[ 9, 8, 11 ], [ 9, 8, 12 ], [ 9, 10, 7 ], [ 9, 10, 8 ], [ 9, 10, 11 ], [ 9, 10, 12 ], [ 9, 11, 7 ], [ 9, 11, 8 ], [ 9, 11, 10 ], [ 9, 11, 12 ], [ 9, 12, 7 ], [ 9, 12, 8 ], [ 9, 12, 10 ], [ 9, 12, 11 ], [ 10, 7, 8 ], [ 10, 7, 9 ], [ 10, 7, 11 ], [ 10, 7, 12 ], [ 10, 8, 7 ], [ 10, 8, 9 ], [ 10, 8, 11 ], [ 10, 8, 12 ], [ 10, 9, 7 ], [ 10, 9, 8 ], [ 10, 9, 11 ], [ 10, 9, 12 ], [ 10, 11, 7 ], [ 10, 11, 8 ], [ 10, 11, 9 ], [ 10, 11, 12 ], [ 10, 12, 7 ], [ 10, 12, 8 ], [ 10, 12, 9 ], [ 10, 12, 11 ], [ 11, 7, 8 ], [ 11, 7, 9 ], [ 11, 7, 10 ], [ 11, 7, 12 ], [ 11, 8, 7 ], [ 11, 8, 9 ], [ 11, 8, 10 ], [ 11, 8, 12 ], [ 11, 9, 7 ], [ 11, 9, 8 ], [ 11, 9, 10 ], [ 11, 9, 12 ], [ 11, 10, 7 ], [ 11, 10, 8 ], [ 11, 10, 9 ], [ 11, 10, 12 ], [ 11, 12, 7 ], [ 11, 12, 8 ],[ 11, 12, 9 ], [ 11, 12, 10 ], [ 12, 7, 8 ], [ 12, 7, 9 ], [ 12, 7, 10 ], [ 12, 7, 11 ], [ 12, 8, 7 ], [ 12, 8, 9 ], [ 12, 8, 10 ], [ 12, 8, 11 ], [ 12, 9, 7 ], [ 12, 9, 8 ], [ 12, 9, 10 ], [ 12, 9, 11 ], [ 12, 10, 7 ], [ 12, 10, 8 ], [ 12, 10, 9 ], [ 12, 10, 11 ], [ 12, 11, 7 ], [ 12, 11, 8 ], [ 12, 11, 9 ], [ 12, 11, 10 ]}and;

$V^{[3]}$ ={[ 13, 14, 15 ], [ 13, 14, 16 ], [ 13, 14, 17 ], [ 13, 14, 18 ], [ 13, 15, 14 ], [ 13, 15, 16 ], [ 13, 15, 17 ], [ 13, 15, 18 ], [ 13, 16, 14 ], [ 13, 16, 15 ],[ 13, 16, 17 ], [ 13, 16, 18 ], [ 13, 17, 14 ], [ 13, 17, 15 ], [ 13, 17, 16 ], [ 13, 17, 18 ], [ 13, 18, 14 ], [ 13, 18, 15 ], [ 13, 18, 16 ], [ 13, 18, 17 ], [ 14, 13, 15 ], [ 14, 13, 16 ], [ 14, 13, 17 ], [ 14, 13, 18 ], [ 14, 15, 13 ], [ 14, 15, 16 ], [ 14, 15, 17 ], [ 14, 15, 18 ], [ 14, 16, 13 ], [ 14, 16, 15 ], [ 14, 16, 17 ], [ 14, 16, 18 ], [ 14, 17, 13 ], [ 14, 17, 15 ], [ 14, 17, 16 ], [ 14, 17, 18 ], [ 14, 18, 13 ], [ 14, 18, 15 ], [ 14, 18, 16 ], [ 14, 18, 17 ], [ 15, 13, 14 ], [ 15, 13, 16 ], [ 15, 13, 17 ], [ 15, 13, 18 ], [ 15, 14, 13 ], [ 15, 14, 16 ], [ 15, 14, 17 ], [ 15, 14, 18 ], [ 15, 16, 13 ], [ 15, 16, 14 ], [ 15, 16, 17 ], [ 15, 16, 18 ], [ 15, 17, 13 ], [ 15, 17, 14 ], [ 15, 17, 16 ], [ 15, 17, 18 ], [ 15, 18, 13 ], [ 15, 18, 14 ], [ 15, 18, 16 ], [ 15, 18, 17 ], [ 16, 13, 14 ], [ 16, 13, 15 ], [ 16, 13, 17 ], [ 16, 13, 18 ], [ 16, 14, 13 ], [ 16, 14, 15 ], [ 16, 14, 17 ], [ 16, 14, 18 ], [ 16, 15, 13 ], [ 16, 15, 14 ], [ 16, 15, 17 ], [ 16, 15, 18 ], [ 16, 17, 13 ], [ 16, 17, 14 ], [ 16, 17, 15 ], [ 16, 17, 18 ], [ 16, 18, 13 ], [ 16, 18, 14 ], [ 16, 18, 15 ], [ 16, 18, 17 ], [ 17, 13, 14 ], [ 17, 13, 15 ], [ 17, 13, 16 ], [ 17, 13, 18 ], [ 17, 14, 13 ], [ 17, 14, 15 ], [ 17, 14, 16 ], [ 17, 14, 18 ], [ 17, 15, 13 ], [ 17, 15, 14 ], [ 17, 15, 16 ], [ 17, 15, 18 ], [ 17, 16, 13 ], [ 17, 16, 14 ], [ 17, 16, 15 ], [ 17, 16, 18 ], [ 17, 18, 13 ], [ 17, 18, 14 ], [ 17, 18, 15 ], [ 17, 18, 16 ], [ 18, 13, 14 ], [ 18, 13, 15 ], [ 18, 13, 16 ], [ 18, 13, 17 ], [ 18, 14, 13 ], [ 18, 14, 15 ], [ 18, 14, 16 ], [ 18, 14, 17 ], [ 18, 15, 13 ], [ 18, 15, 14 ], [ 18, 15, 16 ], [ 18, 15, 17 ], [ 18, 16, 13 ], [ 18, 16, 14 ], [ 18, 16, 15 ], [ 18, 16, 17 ], [ 18, 17, 13 ], [ 18, 17, 14 ], [ 18, 17, 15 ], [ 18, 17, 16 ]}.

The cartesian product of $P^{[3]} \times S^{[3]} \times V^{[3]}$ is generated using the GAP software with $\left|P^{[3]} \times S^{[3]} \times V^{[3]}\right| = 1728000$. G is generated by $< \{(123456), (123)\}, \{(7\ 8\ 9\ 10\ 11\ 12), (789)\}, \{(13\ 14\ 15\ 16\ 17\ 18), (13\ 14\ 15)\} >$ using the GAP software. $([1,2,3], [7,8,9], [13,14,15])$ is fixed by an element $(g_p, g_s, g_v) \in G$ if and only if $1,2$ and $3$ comes from a single cycle in $g_p$ ; $7,8$ and $9$ comes from a single cycle in $g_s$ and $13,14$ and $15$ comes from a single cycle of $g_v$ . The $|Stab_G([1,2,3], [7,8,9], [13,14,15])| = 27$.

By Orbit-Stabilizer Theorem,

$$|Orb_G([1,2,3], [7,8,9], [13,14,15])| = |G: Stab_G([1,2,3], [7,8,9], [13,14,15])|$$
$$= \frac{|G|}{|Stab_G([1,2,3], [7,8,9], [13,14,15])|}$$
$$= \frac{46\ 656\ 000}{27} = 1728000 = \left|P^{[3]} \times S^{[3]} \times V^{[3]}\right|$$

Therefore, $A_6 \times A_6 \times A_6$ acts transitively on $P^{[3]} \times S^{[3]} \times V^{[3]}$.

**Lemma 2.3:** The action of $A_7 \times A_7 \times A_7$ on $P^{[3]} \times S^{[3]} \times V^{[3]}$ is transitive.

**Proof:** Let $G = A_7 \times A_7 \times A_7$ act on $P^{[3]} \times S^{[3]} \times V^{[3]}$ where;

$P^{[3]}$ ={[ 1, 2, 3 ], [ 1, 2, 4 ], [ 1, 2, 5 ], [ 1, 2, 6 ], [ 1, 2, 7 ], [ 1, 3, 2 ], [ 1, 3, 4 ], [ 1, 3, 5 ], [ 1, 3, 6 ], [ 1, 3, 7 ], [ 1, 4, 2 ], [ 1, 4, 3 ], [ 1, 4, 5 ], [ 1, 4, 6 ], [ 1, 4, 7 ], [ 1, 5, 2 ], [ 1, 5, 3 ], [ 1, 5, 4 ], [ 1, 5, 6 ], [ 1, 5, 7 ], [ 1, 6, 2 ], [ 1, 6, 3 ], [ 1, 6, 4 ], [ 1, 6, 5 ], [ 1, 6, 7 ], [ 1, 7, 2 ], [ 1, 7, 3 ], [ 1, 7, 4 ], [ 1, 7, 5 ], [ 1, 7, 6 ], [ 2, 1, 3 ], [ 2, 1, 4 ], [ 2, 1, 5 ], [ 2, 1, 6 ], [ 2, 1, 7 ], [ 2, 3, 1 ], [ 2, 3, 4 ], [ 2, 3, 5 ], [ 2, 3, 6 ], [ 2, 3, 7 ], [ 2, 4, 1 ], [ 2, 4, 3 ], [ 2, 4, 5 ], [ 2, 4, 6 ], [ 2, 4, 7 ], [ 2, 5, 1 ], [ 2, 5, 3 ], [ 2, 5, 4 ], [ 2, 5, 6 ], [ 2, 5, 7 ], [ 2, 6, 1 ], [ 2, 6, 3 ], [ 2, 6, 4 ], [ 2, 6, 5 ], [ 2, 6, 7 ], [ 2, 7, 1 ], [ 2, 7, 3 ], [ 2, 7, 4 ], [ 2, 7, 5 ], [ 2, 7, 6 ], [ 3, 1, 2 ], [ 3, 1, 4 ], [ 3, 1, 5 ], [ 3, 1, 6 ], [ 3, 1, 7 ], [ 3, 2, 1 ], [ 3, 2, 4 ], [ 3, 2, 5 ], [ 3, 2, 6 ], [ 3, 2, 7 ], [ 3, 4, 1 ], [ 3, 4, 2 ], [ 3, 4, 5 ], [ 3, 4, 6 ], [ 3, 4, 7 ], [ 3, 5, 1 ], [ 3, 5, 2 ], [ 3, 5, 4 ], [ 3, 5, 6 ], [ 3, 5, 7 ], [ 3, 6, 1 ], [ 3, 6, 2 ], [ 3, 6, 4 ], [ 3, 6, 5 ], [ 3, 6, 7 ], [ 3, 7, 1 ], [ 3, 7, 2 ], [ 3, 7, 4 ], [ 3, 7, 5 ], [ 3, 7, 6 ], [ 4, 1, 2 ], [ 4, 1, 3 ], [ 4, 1, 5 ], [ 4, 1, 6 ], [ 4, 1, 7 ], [ 4, 2, 1 ],[ 4, 2, 3 ], [ 4, 2, 5 ], [ 4, 2, 6 ], [ 4, 2, 7 ], [ 4, 3, 1 ], [ 4, 3, 2 ], [ 4, 3, 5 ], [ 4, 3, 6 ], [ 4, 3, 7 ], [ 4, 5, 1 ], [ 4, 5, 2 ], [ 4, 5, 3 ], [ 4, 5, 6 ], [ 4, 5, 7 ], [ 4, 6, 1 ], [ 4, 6, 2 ], [ 4, 6, 3 ], [ 4, 6, 5 ], [ 4, 6, 7 ], [ 4, 7, 1 ], [ 4, 7, 2 ], [ 4, 7, 3 ], [ 4, 7, 5 ], [ 4, 7, 6 ], [ 5, 1, 2 ], [ 5, 1, 3 ], [ 5, 1, 4 ], [ 5, 1, 6 ], [ 5, 1, 7 ], [ 5, 2, 1 ], [ 5, 2, 3 ], [ 5, 2, 4 ], [ 5, 2, 6 ], [ 5, 2, 7 ], [ 5, 3, 1 ], [ 5, 3, 2 ], [ 5, 3, 4 ], [ 5, 3, 6 ], [ 5, 3, 7 ], [ 5, 4, 1 ], [ 5, 4, 2 ], [ 5, 4, 3 ], [ 5, 4, 6 ], [ 5, 4, 7 ], [ 5, 6, 1 ], [ 5, 6, 2 ], [ 5, 6, 3 ], [ 5, 6, 4 ], [ 5, 6, 7 ], [ 5, 7, 1 ], [ 5, 7, 2 ], [ 5, 7, 3 ], [ 5, 7, 4 ], [ 5, 7, 6 ], [ 6, 1, 2 ], [ 6, 1, 3 ], [ 6, 1, 4 ], [ 6, 1, 5 ], [ 6, 1, 7 ], [ 6, 2, 1 ], [ 6, 2, 3 ], [ 6, 2, 4 ], [ 6, 2, 5 ], [ 6, 2, 7 ], [ 6, 3, 1 ], [ 6, 3, 2 ], [ 6, 3, 4 ], [ 6, 3, 5 ], [ 6, 3, 7 ], [ 6, 4, 1 ], [ 6, 4, 2 ], [ 6, 4, 3 ], [ 6, 4, 5 ], [ 6, 4, 7 ], [ 6, 5, 1 ], [ 6, 5, 2 ], [ 6, 5, 3 ], [ 6, 5, 4 ], [ 6, 5, 7 ], [ 6, 7, 1 ], [ 6, 7, 2 ], [ 6, 7, 3 ], [ 6, 7, 4 ], [ 6, 7, 5 ], [ 7, 1, 2 ], [ 7, 1, 3 ], [ 7, 1, 4 ], [ 7, 1, 5 ], [ 7, 1, 6 ], [ 7, 2, 1 ], [ 7, 2, 3 ], [ 7, 2, 4 ], [ 7, 2, 5 ], [ 7, 2, 6 ], [ 7, 3, 1 ], [ 7, 3, 2 ], [ 7, 3, 4 ], [ 7, 3, 5 ], [ 7, 3, 6 ], [ 7, 4, 1 ], [ 7, 4, 2 ], [ 7, 4, 3 ], [ 7, 4, 5 ], [ 7, 4, 6 ], [ 7, 5, 1 ], [ 7, 5, 2 ], [ 7, 5, 3 ], [ 7, 5, 4 ], [ 7, 5, 6 ], [ 7, 6, 1 ], [ 7, 6, 2 ], [ 7, 6, 3 ], [ 7, 6, 4 ], [ 7, 6, 5 ]};

$S^{[3]}$ = {[ 8, 9, 10 ], [ 8, 9, 11 ], [ 8, 9, 12 ], [ 8, 9, 13 ], [ 8, 9, 14 ], [ 8, 10, 9 ], [ 8, 10, 11 ], [ 8, 10, 12 ], [ 8, 10, 13 ], [ 8, 10, 14 ], [ 8, 11, 9 ], [ 8, 11, 10 ], [ 8, 11, 12 ], [ 8, 11, 13 ], [ 8, 11, 14 ], [ 8, 12, 9 ], [ 8, 12, 10 ], [ 8, 12, 11 ], [ 8, 12, 13 ], [ 8, 12, 14 ], [ 8, 13, 9 ], [ 8, 13, 10 ], [ 8, 13, 11 ], [ 8, 13, 12 ], [ 8, 13, 14 ], [ 8, 14, 9 ], [ 8, 14, 10 ], [ 8, 14, 11 ], [ 8, 14, 12 ], [ 8, 14, 13 ], [ 9, 8, 10 ], [ 9, 8, 11 ], [ 9, 8, 12 ], [ 9, 8, 13 ], [ 9, 8, 14 ], [ 9, 10, 8 ], [ 9, 10, 11 ], [ 9, 10, 12 ], [ 9, 10, 13 ], [ 9, 10, 14 ], [ 9, 11, 8 ], [ 9, 11, 10 ], [ 9, 11, 12 ], [ 9, 11, 13 ], [ 9, 11, 14 ], [ 9, 12, 8 ], [ 9, 12, 10 ], [ 9, 12, 11 ], [ 9, 12, 13 ], [ 9, 12, 14 ], [ 9, 13, 8 ], [ 9, 13, 10 ], [ 9, 13, 11 ], [ 9, 13, 12 ], [ 9, 13, 14 ], [ 9, 14, 8 ], [ 9, 14, 10 ], [ 9, 14, 11 ], [ 9, 14, 12 ], [ 9, 14, 13 ], [ 10, 8, 9 ], [ 10, 8, 11 ], [ 10, 8, 12 ], [ 10, 8, 13 ], [ 10, 8, 14 ], [ 10, 9, 8 ],[ 10, 9, 11 ], [ 10, 9, 12 ], [ 10, 9, 13 ], [ 10, 9, 14 ], [ 10, 11, 8 ], [ 10, 11, 9 ], [ 10, 11, 12 ], [ 10, 11, 13 ], [ 10, 11, 14 ], [ 10, 12, 8 ], [ 10, 12, 9 ], [ 10, 12, 11 ], [ 10, 12, 13 ], [ 10, 12, 14 ], [ 10, 13, 8 ], [ 10, 13, 9 ], [ 10, 13, 11 ], [ 10, 13, 12 ], [ 10, 13, 14 ], [ 10, 14, 8 ], [ 10, 14, 9 ], [ 10, 14, 11 ], [ 10, 14, 12 ], [ 10, 14, 13 ], [ 11, 8, 9 ], [ 11, 8, 10 ], [ 11, 8, 12 ], [ 11, 8, 13 ], [ 11, 8, 14 ], [ 11, 9, 8 ], [ 11, 9, 10 ], [ 11, 9, 12 ], [ 11, 9, 13 ], [ 11, 9, 14 ], [ 11, 10, 8 ], [ 11, 10, 9 ], [ 11, 10, 12 ], [ 11, 10, 13 ], [ 11, 10, 14 ], [ 11, 12, 8 ], [ 11, 12, 9 ], [ 11, 12, 10 ], [ 11, 12, 13 ], [ 11, 12, 14 ], [ 11, 13, 8 ], [ 11, 13, 9 ], [ 11, 13, 10 ], [ 11, 13, 12 ], [ 11, 13, 14 ], [ 11, 14, 8 ], [ 11, 14, 9 ], [ 11, 14, 10 ], [ 11, 14, 12 ], [ 11, 14, 13 ], [ 12, 8, 9 ], [ 12, 8, 10 ], [ 12, 8, 11 ], [ 12, 8, 13 ], [ 12, 8, 14 ], [ 12, 9, 8 ], [ 12, 9, 10 ], [ 12, 9, 11 ], [ 12, 9, 13 ], [ 12, 9, 14 ], [ 12, 10, 8 ], [ 12, 10, 9 ],[ 12, 10, 11 ], [ 12, 10, 13 ], [ 12, 10, 14 ], [ 12, 11, 8 ], [ 12, 11, 9 ], [ 12, 11, 10 ], [ 12, 11, 13 ], [ 12, 11, 14 ], [ 12, 13, 8 ], [ 12, 13, 9 ], [ 12, 13, 10 ], [ 12, 13, 11 ], [ 12, 13, 14 ], [ 12, 14, 8 ], [ 12, 14, 9 ], [ 12, 14, 10 ], [ 12, 14, 11 ], [ 12, 14, 13 ], [ 13, 8, 9 ], [ 13, 8, 10 ], [ 13, 8, 11 ], [ 13, 8, 12 ], [ 13, 8, 14 ], [ 13, 9, 8 ], [ 13, 9, 10 ], [ 13, 9, 11 ], [ 13, 9, 12 ], [ 13, 9, 14 ], [ 13, 10, 8 ], [ 13, 10, 9 ], [ 13, 10, 11 ], [ 13, 10, 12 ],[ 13, 10, 14 ], [ 13, 11, 8 ], [ 13, 11, 9 ], [ 13, 11, 10 ], [ 13, 11, 12 ], [ 13, 11, 14 ], [ 13, 12, 8 ], [ 13, 12, 9 ], [ 13, 12, 10 ], [ 13, 12, 11 ], [ 13, 12, 14 ], [ 13, 14, 8 ], [ 13, 14, 9 ], [ 13, 14, 10 ], [ 13, 14, 11 ], [ 13, 14, 12 ], [ 14, 8, 9 ], [ 14, 8, 10 ], [ 14, 8, 11 ], [ 14, 8, 12 ], [ 14, 8, 13 ], [ 14, 9, 8 ], [ 14, 9, 10 ], [ 14, 9, 11 ], [ 14, 9, 12 ], [ 14, 9, 13 ], [ 14, 10, 8 ], [ 14, 10, 9 ], [ 14, 10, 11 ], [ 14, 10, 12 ], [ 14, 10, 13 ], [ 14, 11, 8 ], [ 14, 11, 9 ], [ 14, 11, 10 ], [ 14, 11, 12 ], [ 14, 11, 13 ], [ 14, 12, 8 ], [ 14, 12, 9 ], [ 14, 12, 10 ], [ 14, 12, 11 ], [ 14, 12, 13 ], [ 14, 13, 8 ], [ 14, 13, 9 ], [ 14, 13, 10 ], [ 14, 13, 11 ], [ 14, 13, 12 ]};

$V^{[3]}$ ={[ 15, 16, 17 ], [ 15, 16, 18 ], [ 15, 16, 19 ], [ 15, 16, 20 ], [ 15, 16, 21 ], [ 15, 17, 16 ], [ 15, 17, 18 ], [ 15, 17, 19 ], [ 15, 17, 20 ], [ 15, 17, 21 ], [ 15, 18, 16 ], [ 15, 18, 17 ], [ 15, 18, 19 ], [ 15, 18, 20 ], [ 15, 18, 21 ], [ 15, 19, 16 ], [ 15, 19, 17 ], [ 15, 19, 18 ], [ 15, 19, 20 ], [ 15, 19, 21 ], [ 15, 20, 16 ], [ 15, 20, 17 ], [ 15, 20, 18 ], [ 15, 20, 19 ], [ 15, 20, 21 ], [ 15, 21, 16 ], [ 15, 21, 17 ], [ 15, 21, 18 ], [ 15, 21, 19 ], [ 15, 21, 20 ], [ 16, 15, 17 ], [ 16, 15, 18 ], [ 16, 15, 19 ], [ 16, 15, 20 ], [ 16, 15, 21 ], [ 16, 17, 15 ], [ 16, 17, 18 ], [ 16, 17, 19 ], [ 16, 17, 20 ], [ 16, 17, 21 ], [ 16, 18, 15 ], [ 16, 18, 17 ], [ 16, 18, 19 ], [ 16, 18, 20 ], [ 16, 18, 21 ], [ 16, 19, 15 ], [ 16, 19, 17 ], [ 16, 19, 18 ], [ 16, 19, 20 ], [ 16, 19, 21 ], [ 16, 20, 15 ], [ 16, 20, 17 ], [ 16, 20, 18 ], [ 16, 20, 19 ], [ 16, 20, 21 ], [ 16, 21, 15 ], [ 16, 21, 17 ], [ 16, 21, 18 ], [ 16, 21, 19 ], [ 16, 21, 20 ], [ 17, 15, 16 ], [ 17, 15, 18 ], [ 17, 15, 19 ], [ 17, 15, 20 ], [ 17, 15, 21 ], [ 17, 16, 15 ], [ 17, 16, 18 ], [ 17, 16, 19 ], [ 17, 16, 20 ], [ 17, 16, 21 ], [ 17, 18, 15 ], [ 17, 18, 16 ], [ 17, 18, 19 ], [ 17, 18, 20 ], [ 17, 18, 21 ], [ 17, 19, 15 ], [ 17, 19, 16 ], [ 17, 19, 18 ], [ 17, 19, 20 ], [ 17, 19, 21 ], [ 17, 20, 15 ], [ 17, 20, 16 ], [ 17, 20, 18 ], [ 17, 20, 19 ], [ 17, 20, 21 ], [ 17, 21, 15 ], [ 17, 21, 16 ], [ 17, 21, 18 ], [ 17, 21, 19 ], [ 17, 21, 20 ], [ 18, 15, 16 ], [ 18, 15, 17 ], [ 18, 15, 19 ], [ 18, 15, 20 ], [ 18, 15, 21 ], [ 18, 16, 15 ], [ 18, 16, 17 ], [ 18, 16, 19 ], [ 18, 16, 20 ], [ 18, 16, 21 ], [ 18, 17, 15 ], [ 18, 17, 16 ], [ 18, 17, 19 ], [ 18, 17, 20 ], [ 18, 17, 21 ], [ 18, 19, 15 ], [ 18, 19, 16 ], [ 18, 19, 17 ], [ 18, 19, 20 ], [ 18, 19, 21 ], [ 18, 20, 15 ], [ 18, 20, 16 ], [ 18, 20, 17 ], [ 18, 20, 19 ], [ 18, 20, 21 ], [ 18, 21, 15 ], [ 18, 21, 16 ], [ 18, 21, 17 ], [ 18, 21, 19 ], [ 18, 21, 20 ], [ 19, 15, 16 ], [ 19, 15, 17 ], [ 19, 15, 18 ], [ 19, 15, 20 ], [ 19, 15, 21 ], [ 19, 16, 15 ], [ 19, 16, 17 ], [ 19, 16, 18 ], [ 19, 16, 20 ], [ 19, 16, 21 ], [ 19, 17, 15 ], [ 19, 17, 16 ], [ 19, 17, 18 ], [ 19, 17, 20 ], [ 19, 17, 21 ], [ 19, 18, 15 ], [ 19, 18, 16 ], [ 19, 18, 17 ], [ 19, 18, 20 ], [ 19, 18, 21 ], [ 19, 20, 15 ], [ 19, 20, 16 ], [ 19, 20, 17 ], [ 19, 20, 18 ], [ 19, 20, 21 ], [ 19, 21, 15 ], [ 19, 21, 16 ], [ 19, 21, 17 ], [ 19, 21, 18 ], [ 19, 21, 20 ], [ 20, 15, 16 ], [ 20, 15, 17 ], [ 20, 15, 18 ], [ 20, 15, 19 ], [ 20, 15, 21 ], [ 20, 16, 15 ], [ 20, 16, 17 ], [ 20, 16, 18 ], [ 20, 16, 19 ], [ 20, 16, 21 ], [ 20, 17, 15 ], [ 20, 17, 16 ], [ 20, 17, 18 ], [ 20, 17, 19 ], [ 20, 17, 21 ], [ 20, 18, 15 ], [ 20, 18, 16 ], [ 20, 18, 17 ], [ 20, 18, 19 ], [ 20, 18, 21 ], [ 20, 19, 15 ], [ 20, 19, 16 ], [ 20, 19, 17 ], [ 20, 19, 18 ], [ 20, 19, 21 ], [ 20, 21, 15 ], [ 20, 21, 16 ], [ 20, 21, 17 ], [ 20, 21, 18 ], [ 20, 21, 19 ], [ 21, 15, 16 ], [ 21, 15, 17 ], [ 21, 15, 18 ], [ 21, 15, 19 ], [ 21, 15, 20 ], [ 21, 16, 15 ], [ 21, 16, 17 ], [ 21, 16, 18 ], [ 21, 16, 19 ], [ 21, 16, 20 ], [ 21, 17, 15 ], [ 21, 17, 16 ], [ 21, 17, 18 ], [ 21, 17, 19 ], [ 21, 17, 20 ], [ 21, 18, 15 ], [ 21, 18, 16 ], [ 21, 18, 17 ], [ 21, 18, 19 ], [ 21, 18, 20 ], [ 21, 19, 15 ], [ 21, 19, 16 ], [ 21, 19, 17 ], [ 21, 19, 18 ], [ 21, 19, 20 ], [ 21, 20, 15 ], [ 21, 20, 16 ], [ 21, 20, 17 ], [ 21, 20, 18 ], [ 21, 20, 19 ]}.

The Cartesian product of $P^{[3]} \times S^{[3]} \times V^{[3]}$ is generated using the GAP software with $\left|P^{[3]} \times S^{[3]} \times V^{[3]}\right| = 9\,261\,000$. G is generated by $< \{(1234567), (123)\}, \{(8\ 9\ 10\ 11\ 12\ 13\ 14), (8\ 9\ 10)\}, \{(15\ 16\ 17\ 18\ 19\ 20\ 21), (15\ 16\ 17)\} >$ using the GAP software. $([1,2,3], [8,9,10], [15,16,17])$ is fixed by an element $(g_p, g_s, g_v) \in G$ if and only if $1,2$ and $3$ comes from a single cycle of $g_p$ ; $8,9$ and $10$ comes from a single cycle of $g_s$ and $15,16$ and $17$ comes from a single cycle of $g_v$ .
The $|Stab_G\ ([1,2,3], [8,9,10], [15,16,17])| = 1728$.

By Orbit-Stabilizer Theorem,
$$|Orb_G\ ([1,2,3], [8,9,10], [15,16,17])| = |G: Stab_G\ ([1,2,3], [8,9,10], [15,16,17])|$$
$$= \frac{|G|}{|Stab_G\ ([1,2,3], [8,9,10], [15,16,17])|}$$
$$= \frac{16\ 003\ 008\ 000}{1728} = 9261000 = \left|P^{[3]} \times S^{[3]} \times V^{[3]}\right|$$
Therefore, $A_7 \times A_7 \times A_7$ acts transitively on $P^{[3]} \times S^{[3]} \times V^{[3]}$ .

**Theorem 2.4:** The action of $A_n \times A_n \times A_n$ on $P^{[3]} \times S^{[3]} \times V^{[3]}$ is transitive if and only if $n \geq 5$.

**Proof:** Let $G = G_p \times G_s \times G_v = A_n \times A_n \times A_n$ act on $P^{[3]} \times S^{[3]} \times V^{[3]}$. It suffices to verify that $\left|P^{[3]} \times S^{[3]} \times V^{[3]}\right|$ is equal to $|Orb_G\ ([1,2,3], [n+1, n+2, n+3], [2n+1, 2n+2, 2n+3])|$ .
Let $|R| = |Stab_G([1,2,3], [n+1, n+2, n+3], [2n+1, 2n+2, 2n+3])|$.
So, $(g_p, g_s, g_v) \in G = A_n \times A_n \times A_n$ fixes
$([1,2,3], [n+1, n+2, n+3], [2n+1, 2n+2, 2n+3]) \in P^{[3]} \times S^{[3]} \times V^{[3]}$ if and only if $1,2$ and $3$ comes from $1$-cycle of $g_p$ ; $n+1, n+2$ and $n+3$ comes from $1$-cycle of $g_s$ and $2n+1, 2n+2$ and $2n+3$ comes from $1$-cycle of $g_v$ .
The $Stab_G([1,2,3], [n+1, n+2, n+3], [2n+1, 2n+2, 2n+3])$ is isomorphic to: $A_{n-3} \times A_{n-3} \times A_{n-3}$ .

Therefore,
$$|R| = |Stab_G([1,2,3], [n+1, n+2, n+3], [2n+1, 2n+2, 2n+3])| = \left|Stab_{G_p}([1,2,3]) \times Stab_{G_s}([n+1, n+2, n+3]) \times Stab_{G_v}([2n+1, 2n+2, 2n+3])\right|$$
$$|R| = \frac{(n-3)! \times (n-3)! \times (n-3)!}{2 \times 2 \times 2} = \left(\frac{(n-3)!}{2}\right)^3$$

Applying the Orbit-Stabilizer Theorem we get;
$$|Orb_G\ ([1,2,3], [n+1, n+2, n+3], [2n+1, 2n+2, 2n+3])|$$
$$= |G: Stab_G([1,2,3], [n+1, n+2, n+3], [2n+1, 2n+2, 2n+3])|$$
$$|G| = \frac{n! \times n! \times n!}{2 \times 2 \times 2} = \left(\frac{n!}{2}\right)^3. \qquad \frac{|G|}{|R|} = \frac{\left(\frac{n!}{2}\right)^3}{\left(\frac{(n-3)!}{2}\right)^3} = \left(\frac{n!}{(n-3)!}\right)^3.$$
Therefore;
$$\frac{|G|}{|R|} = \left(\frac{n!}{(n-3)!}\right)^3 = \left|P^{[3]} \times S^{[3]} \times V^{[3]}\right|$$
Hence, $A_n \times A_n \times A_n$ acts transitively on $P^{[3]} \times S^{[3]} \times V^{[3]}$ if $n \geq 5$.

**Corollary 2.5:** For $n < 5$, the $|Stab_G([1,2,3], [n+1, n+2, n+3], [2n+1, 2n+2, 2n+3])| = |A_{n-3} \times A_{n-3} \times A_{n-3}| < 1$.

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