CHUKA UNIVERSITY

UNIVERSITY EXAMINATIONS

THIRD YEAR EXAMINATION FOR THE AWARD OF DEGREE OF BACHELOR OF EDUCATION (SCIENCE, ARTS) BACHELOR OF SCIENCE (GENERAL) BACHELOR OF SCIENCE (MATHEMATICS) , BACHELOR OF ARTS (MATHS ,ECONS)

MATH 303: REAL ANALYSIS II

STREAMS: **TIME: 2 HOURS**

DAY/DATE: WEDNESDAY 12/04/2023 **2.30 P.M – 4.30 P.M**

INSTRUCTIONS:

- Answer question **ONE** and **TWO** other questions
- Do not write on the question paper
- Write your answers legibly and use your time wisely

QUESTION ONE: (30 MARKS)

(a) Discuss the following concepts as used in analysis

i. A partion of a closed interval $[a, b]$ (1 mark)

ii. The Riemanns' upper sum and lower sum of the function f (3 marks)

iii. The Riemann Stieltjes integrable function on $[a, b]$ (3 marks)

(b) Prove that if $f_n \neq 0 \;\; \forall\, n \in N$ is a sequence of functions which converges to the limit $f \neq 0$ then $\frac{1}{f_n}$ is convergent to $\frac{1}{f}$ $(as\; n \to \infty)$ (5 marks)

(c) Define a Step function. Hence show that a step function is always Riemann Integrable. (4 marks)

(d) By sketching the graphs of the function $f(x) = log_5 x$ and $f(x) = log_{\frac{1}{5}} x$ on the same axis, state the relationship between the two graphs (5 marks)

(e) Let $f(x) = x \ for \ a \leq x \leq b$ and define $\propto on [a,b]$ by $\propto x = 0$ for $a \leq x \leq b$ with $\propto (b) = c$. If (P,t) is a tagged partition of $[a,b]$with $P = \{x_0, x_1, x_2, \dots x_n\}$, define $S(P,t,f,\propto) = t_n c$. Show that $\int_a^b xd \propto (x) = bc$ (3 marks)

(f) Distinguish between pointwise convergence and uniform convergence of sequences of functions (2 marks)

(g) Distinguish an odd function and an even function. Give an example in each case. Hence prove that if the function $f(x)$ is even and $g(x)$ is odd, then the product $f(x)g(x)$ is odd (4 marks)

QUESTION TWO: (20 MARKS)

(a) State and prove the Comparison test (Weierstrauss M-test) for convergence of real valued infinite series of functions. (9 marks)

(b) Hence by using Comparison theorem, prove that for a series of positive terms $\sum_{n\in N} f_n$ such that $\lim_{n\to\infty}\left(\frac{f_{n+1}}{f_n}\right) = l$, if $l < 1$ then series is convergent (6 marks)

(a) (i) State the Cauchy criterion for convergence of a series of functions (2 marks)

(ii) Hence prove that for a series of functions $\sum_{n\in N} f_n$ on **K,** this series only converges if $\forall \ \varepsilon > 0 \ \exists \ N(\varepsilon) \in \boldsymbol{N}: |\sum_{k=m}^{n} f_n| < \varepsilon \ \ for \ every \ n \geq m \geq N(\varepsilon)$ (3 marks)

QUESTION THREE: (20 MARKS)

(a) Describe in details how each of the following integrals are used to determine the area of the function f on the interval $[a,b]$

(i) Riemann Integral (4 marks)

(ii) Riemann-Stieltjes Integral (3 marks)

(c) Illustrate whether a Dirichlet function on the interval $[a,b]$ is Riemann Integrable or not. (4 marks)

(d) Show that the function $f(x) = 5x$ is Riemann integrable on $[0,1]$ and that $\int_0^1 f(x) = 2.5$ (9 marks)

QUESTION FOUR: (20 MARKS)

(a) Define the Fourier series of the function $f(x)$ on the interval $-l \ to \ l$. (3 marks)

(b) Let $f(x)$ be a function of period 2π such that

$$f(x) = \begin{cases} 1 & -\pi < x < 0 \\ 0 & 0 < x < \pi \end{cases}$$

(i) Sketch a graph of $f(x)$ in the interval $-2\pi < x < 2\pi$ (2 marks)

(ii) Show that the Fourier series for $f(x)$ in the interval $-\pi < x < \pi$ is

$$\frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3} sin3x + \frac{1}{5} sin5x + \cdots\right]$$

(8 marks)

(c) State and prove the Cauchy's Root Test for convergence of functions of an infinite series. (7 marks)

<u>QUESTION FIVE: (20 MARKS)</u>

(a) Define an absolutely convergent series. Hence prove that an absolute convergent series of functions in $(\boldsymbol{K}, d)$ is necessarily convergent, however using an appropriate example illustrate that the converse of this is not true. (8 marks)

(b) State and prove the Intermediate Mean Value Theorem (6 marks)

(c) Let $f: [p, q] \rightarrow R$ be continuous and that $f(p)f(q) < 0$, then there exists $c\ \epsilon (p, q)$ such that $f(c) = 0$. (6 marks)
