CHUKA UNIVERSITY

UNIVERSITY EXAMINATIONS

FOURTH YEAR EXAMINATION FOR THE DEGREE OF BACHELORS OF EDUCATION SCIENCE/ARTS; BACHELORS OF SCIENCE MATHEMATICS

MATH 403: MEASURE THEORY

STREAMS: ````as above```` Y4S2 **TIME: 2HRS**

INSTRUCTIONS:

- Answer question **ONE** and **TWO** other questions

QUESTION ONE: (30 MARKS)

a) Prove that a sigma algebra is closed under countable intersections. (4 mks)

b) Prove the following properties of an outer measure $\mu*$

i. $\mu^*(\phi) = 0$ (2 mks)

ii. $\mu^*(\{x\}) = 0$ (2 mks)

iii. If $A \subseteq B$ and A is measurable of finite measure, then $\mu*(B - A) = \mu*(B) - \mu*(A)$ (4 mks)

c) Let (X, x, μ) be a measure space and $A\epsilon x$, define$\tau(E) = \mu(E \cap A)$ for every $E\epsilon x$. Show that τ is a measure. (4mks)

d) Prove that if $\mu*(A) = 0$, then $\mu*(A \cup B) = \mu*(B)$ (3 mks)

e) Define a simple function. Hence find the integral of the simple function defined by

$$f(x) = \begin{cases} 1 & if \quad x \in C \cap \{rationals\} \\ 2 & if \quad x \in C \cap \{irrationals\} \\ 3 & if \quad x \in [01] - C \end{cases}$$

Where C is the cantor set (3mks)

f) Let f and g be non-negative extended real valued measurable functions both defined on sets E and F such that $E \subseteq F$, both measurable sets, show that

i. $\int_E f d\mu \leq \int_E g d\mu$ if $f \leq g$ (3mks)

ii. $\int_E f d\mu \leq \int_F f d\mu$ (3 mks)

QUESTION TWO: (20 MARKS)

a) Given that the sets A and B are Lebesgue measurable sets, prove that
   i. $A \cup B$ is Lebesgue measurable
   ii. $A \cap B$ is Lebesgue measurable (6 mks)

b) Let X be a non empty set and $\{\mathfrak{x}_i\}$ be a collection of sigma algebras of subsets of X. prove that $\cap\, \mathfrak{x}_i$ is a sigma algebra (7 mks)

c) Let X=$\mathbb{R}$, define $\mathscr{B}$=$\{B \subseteq \mathbb{R}: B\ is\ countable\ or\ \mathbb{R} - B\ is\ countable\}$. Prove that $\mathscr{B}$ is a sigma algebra (7 mks)

QUESTION THREE: (20 MARKS)

a) Let $(X, \mathfrak{x})$ be a measurable space and f: X → $\mathbb{R}^*$ be a given function. Show that the following statements are equivalent

   i. $\{x \in X: f(x) \geq a\} = f^{-1}[a, \infty] \in \mathfrak{x}$ for all $a \in \mathbb{R}^*$

   ii. $\{x \in X: f(x) < a\} = f^{-1}[-\infty, a) \in \mathfrak{x}$ for all $a \in \mathbb{R}^*$

   iii. $\{x \in X: f(x) \leq a\} = f^{-1}[-\infty, a] \in \mathfrak{x}$ for all $a \in \mathbb{R}^*$

   iv. $\{x \in X: f(x) > a\} = f^{-1}(a, \infty] \in \mathfrak{x}$ for all $a \in \mathbb{R}^*$ (8 mks)

c) Let $(X, \mathfrak{x})$ be a measurable space and f: X → $\mathbb{R}^*$ be $\mathfrak{x}$ measurable. The prove that

   (i)$f^2 = f^2(x)\ \ \forall x \in X$ is also $\mathfrak{x}$ measurable.
   (ii) $|f| = |f(x)|\ \ \forall x \in X$ is also $\mathfrak{x}$ measurable
   Using an appropriate counter example show that the converse of (i) is not necessarily true (8 mks)

d) Suppose f and g are extended real valued functions defined on a measurable set E.

Show that if f is Lebesgue measurable on E, and g=f measure almost everywhere , then g is Lebesgue measurable

(6 mks)

QUESTION FOUR: (20 MARKS)

a) State without prove the monotone convergence theorem (M.C.T) (2 mks)

b) (i)Explain a uniform convergence sequence of functions (2 mks)

   (ii) Show that M.C.T does not apply in the sequence $f_n(x) = \frac{1}{n}\chi_{[0,n]}$ for $n \in N$. Explain your answer. (4 mks)

   (iii) Verify whether or not Fatous lemma applies for (ii) above (2 mks)

c) Show that the set of measurable sets form a sigma algebra (10 mks)

QUESTION FIVE: (20 MARKS)

a) Show that the collection of open intervals in $\mathbb{R}$ do not form a sigma algebra (2 mks)

b) **(i)** By using the properties of outer measure, show that the unit interval [0,1] is not countable. (2mks)

(ii) Prove or disapprove that every set that has outer measure zero is countable (2 mks)

(iii) Let A be the set of irrational numbers in the unit interval. Prove that $\mu*(A)=1$ (3 mks)

c) Define a Lebesgue non-measurable set. Prove that if E is non-Lebesque measurable subset of $\mathbb{R}$, then there exists a subset $A$of Esuch that $0 < \mathcal{M}^*[A] < \infty$ (5 mks)

d) Define a ‘’totally unlucky number’’ to be a number that does not have the digit 7 in its decimal expansion. Calculate the measure of all such numbers in the unit interval [0,1] (6 mks)