CHUKA UNIVERSITY

UNIVERSITY EXAMINATIONS

FOURTH YEAR EXAMINATION FOR THE AWARD OF DEGREE OF BACHELOR OF SCIENCE IN MATHEMATICS, BACHELOR OF EDUCATION

MATH 409: FUNCTIONAL ANALYSIS

STREAMS: BSc. Maths; Bed Sc/Arts, BA. Maths (Y4S2) TIME: 2 HOURS

DAY/DATE: TUESDAY 11/04/2023 11.30 A.M. – 1.30 P.M.

INSTRUCTIONS:

- **Answer question ONE and TWO other questions**
- **Sketch maps and diagrams may be used whenever they help to illustrate your answer**
- **Do not write on the question paper**
- **Write your answers legibly and use your time wisely**

QUESTION ONE (30 MARKS)

(a) Distinguish the following terms as used in functional analysis

i. A convergent sequence and a Cauchy sequence (4 marks)

ii. Holders Inequality and Minikowsk`s Inequality (4 marks)

iii. An Iteration and a contraction mapping (4 marks)

iv. A complete space and a compact space (2 marks)

(b) Let $\|.\|_0$ and $\|.\|_1$ be two norms on a vector space. When are these two norms said to be equivalent? (2 marks)

(c) Define an orthonormal set. Hence show that an orthonormal set is linearly Independent (3 marks)

(d) (i) Define a sesquillinear functional on normed linear spaces $X\ and\ Y$ (2 marks)
(ii) Hence state without proof the Riesz's Representation Theorem (3 marks)

(e) Let $T: X \rightarrow Y$ be a linear operator from a normed linear space X into a normed linear space Y, prove that if T is continuous at the origin then it is uniformly continuous on X. (3 marks)

(f) Prove that if the limit point of a weakly convergent sequence exists, then that limit is unique (3 marks)

QUESTION TWO: (20 MARKS)

(a) Distinguish between a Hamel base and Schauder Basis. Hence illustrate that in finite dimensional spaces Schauder Basis and Hamel base coincide (6 marks)

(b) By defining a semi-norm and a para-norm on a linear space X, show that every semi-norm is a para-norm but the converse is not true (8 marks)

(c) When are two normed linear spaces said to be Isometrically Isomorphic? Hence show that the spaces $C_O{}^* \sim \ell_1$ are Isometrically Isomorphic (6 marks)

QUESTION THREE: (20 MARKS)

(a) Distinguish between a Banach space and a Hilbert space. Hence using appropriate examples illustrate Hilbert spaces are necessarily Banach spaces however, the converse is not true. (8 marks)

(b) Prove that the mapping $\|.\|: R^n \to R$ defined by

$\| x \|_\infty = \left((\sum_{k=1}^{n} |x_k|^2) \right)^{\frac{1}{2}}$ is a norm. (7 marks)

(c) Prove that a sequence of real numbers converges if and only if it is a Cauchy sequence (5 marks)

QUESTION FOUR: (20 MARKS)

(a) State and prove the Banach Fixed Point Theorem (the Contraction Mapping Theorem) in a metric space X. (10 marks)

(b) Prove that strong convergence in sequences implies weak convergence but the converse is not necessarily true (10 marks)

QUESTION FIVE: (20 MARKS)

(a) Let $T: X \to Y$ be a linear operator from a normed linear space X into a normed linear space Y. Prove that T is continuous if and only if T is bounded. (6 marks)

(b) (i) Define a Uniformly bounded set function f on normed linear spaces X, Y. (2 marks)

(ii) Hence state and prove the Uniform Bounded Theorem (7 marks)

(c) Define $d: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\} \quad x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R}^2$$

Show that d is a metric on $\mathbb{R}^2$ (5 marks)
