FORMULATION OF SCHRÖDINGER EQUATION USING THE HILBERT SPACE OPERATORS APPROACH

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DECLARATION AND RECOMMENDATION

Declaration

This thesis is my original work and has not been presented for an award of a degree in any other University.

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Recommendation

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DEDICATION

I dedicate this project to my mother Angeline, my husband Alphonce and my daughters Victoria and Meghan.

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ABSTRACT

Operators in Hilbert space have properties which are useful in the study of mathematical abstract areas such as approximation theory, Banach Fixed point theory, the spectral theory as well as Quantum Mechanics. Schrödinger equation is a fundamental entity with many applications in Quantum Mechanics. This equation was initially derived by applying the knowledge of electromagnetic wave function and Einstein theory of relativity. Later, it was derived by applying the knowledge of Newtonian mechanics. It was also derived by extending the wave equation for classical fields to photons and simplified using approximations consistent with generalized non-zero rest mass. However, from the existing literature no study has been done on deriving Schrödinger equation using properties of Hilbert space operators. In this study, Hilbert space operators that include unitary operators, self adjoint operators and compact operators, norms of linear operators, Hilbert Schmidt operator, normal operators together with Lebesque Integral, Neumann Integral and spectrum are used in place of the existing concepts of electromagnetic wave function, Einstein theory of relativity and approximation consistent with generalized non zero mass to derive the Schrödinger equation. Furthermore, this study has established the correlation between the electromagnetic wave function and Einstein theory of relativity in relation with Hilbert space operators. Application of Hilbert space operators on Quantum observables such as position, momentum and energy of a particle has been done in these study. The derivation of Schrödinger enhances equation and its application using Hilbert space operators have enhanced a better understanding of the concept of Schrödinger equation. The results of this work will be useful in quantum mechanics as well as in mathematical operator theory.

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CHAPTER ONE
INTRODUCTION

1.1 Background Information

Schrödinger equation was first derived by Schrödinger in 1926. In his work he used the knowledge of electromagnetic prototype of wave equation $(\nu^2 \bigtriangledown^2 - \frac{d^2}{dt^2})E = 0$ and Einstein equation $E = mc^2$ (Ward and Volkmer, 2006). The purpose of his study was to find the wave function of the electron. Nelson (1966) used Newtonian mechanics to derive Schrödinger equation. In his work he used the hypothesis that any particle of mass *m* constantly undergoes Brownian motion with diffusion coefficient $\frac{\hbar}{2m}$. Ward and Volkmer (2006) derived Schrödinger equation by extending the wave equation for classical fields to photons and generalized to non-zero rest mass particles and using approximations consistent with non-relativistic particles.

Hilbert space gives a means by which one can consider functions as points belonging to an infinite dimensional space. Leversha (2010) stated that, the states of quantum systems are identified by unit vectors in an infinite dimensional complex Hilbert space and observables such as position, momentum and energy are realised as self-adjoint linear operators acting on the space. Consequently, Leversha (2010) showed the relationship between the needs of physics and the mathematics of operators of Hilbert spaces.

Mathematical techniques are widely applied in quantum mechanics such as in finding the probability position of a particle at a particular place. Gagne (2013) stated that if the system S in a state lying $S \subset M$, i.e. $x \in S$, the characteristic function $X_s(x) = 1$ while if $x \notin S$ then $X_s(x) = 0$ for$M$ is the phase state. Gagne (2013) showed that the probability density of a particle is given by $pr(\Psi, x \in \Delta)$ where Δ is the probability region where a particle can be found i.e.

$$\int_{\mathbb{R}^3} \|\Psi(x)\|^2 dx = 1 \tag{1.1}$$

Packel (1974) showed that, Hilbert space axioms are a most natural generalization of properties of finite dimensional Euclidean space. In his work, Packel stated that the theory of orthonormal bases and spectral decomposition grows directly out of Hilbert's work with quadratic forms and integral equations. Significantly, Hilbert space gives a natural and effective setting for one formulation of quantum mechanics. This Quantum mechanics, Leversha (2010) stated that the physical theory that describes the behavior of matter in the microscopic realm and hence reigns supreme in the realm of molecules, atoms and subatomic particles.

According to Mostafazadeh (2004), Hilbert space with time dependent inner product arises to develop a non-relativistic quantum mechanics of a particle confined into move on an oscillating membrane. In his work Mostafazadeh (2004) stated that, a linear operator
$U : \mathcal{H}_1 \to \mathcal{H}_2$ where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces is said to be *unitary* operator if $\Psi, \Phi \in \mathcal{H}_1, \langle U\Psi, U\Phi\rangle_2 = \langle \Psi, \Phi\rangle_1$ where $\langle ., .\rangle_1$ and $\langle ., .\rangle_2$ respectively stand for inner product of $\mathcal{H}_1$ and $\mathcal{H}_2$. Mostafazadeh (2004) found that the physical observables are identified with Hermitian operators $O : \mathcal{H} \to \mathcal{H}$ and dynamics governed by Schrödinger equation.

$$i\hbar\frac{\partial}{\partial t}\Psi(r,t) = H\Psi(r,t) \tag{1.2}$$

where i is imaginary unit, $\hbar$ reduced Planck constant and H Hamiltonian and may be time dependent. Heslot (1985) stated that the motion of a Quantum system is governed by a Schrödinger equation $i\hbar\frac{d}{dt}|\Psi\rangle = \hat{H}|\Psi\rangle$. Assuming that$|\ \Psi\rangle$ is normalized, then the Hamiltonian function H can be defined in terms of $\hat{H}$, Hamiltonian operator as $\langle\Psi \mid \hat{H} \mid \Psi\rangle$

Siddiqi and Nanda (1986) stated that, Hilbert space was first introduced by David Hilbert between 1862- 1943. In their work they stated that Hilbert space, is complete inner product space. Siddiqi and Nanda (1986)stated that, If T is an operator on Hilbert space $\mathcal{H}$ then:

(i) T is normal if $TT^\star = T^\star T$

(ii) T is self-adjoint (or Hermitian) if $T = T^\star$

(iii) $T \in B(\mathcal{H})$ is positive if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$

(iv) T is unitary if $TT^\star = T^\star T = I$

Grigoriu (1995) stated that, Let $T \in B(\mathcal{H})$ on Hilbert space, by Reisz Representation theorem, there exists a unique vector $z = z_y \in H$ so that $\langle y, Tx\rangle = \langle z_y, x\rangle$ for all $x \in H$. The map $T^* : \mathcal{H} \to \mathcal{H}$ is defined as $T^* y = z_y$. Thus $\langle T^* y, x\rangle = \langle y, Tx\rangle\ \forall x, y \in \mathcal{H}$ which uniquely determines $T^\star y$ for $y \in \mathcal{H}$. Thus T^* is an adjoint operator of T . If two elements of the set M

are pairwise orthogonal vectors, each of the vector is normalized and each has a norm equal to one, then a set M is called orthonormal (Akhiezer and Glazman, 2013).

The definition of Riemann's integral is adopted from Trench (2012). Let f be defined on $[a, b]$, then Trench stated that f is said to be *Riemann integrable* on $[a, b]$ if there is a number L with the following property. For every $\epsilon>0$ there exists a $\delta>0$ such that $|\sigma - L| < \epsilon$. If σ is Riemann's sum of f over partition P of $[a, b]$ such that $\|P\| < \delta$. Then L is Riemann's integral of f over

$$\int_a^b f(x)dx = L. \tag{1.3}$$

Suppose $\mathcal{H}$ is a separable Hilbert space and $T \in B(\mathcal{H})$. According to Al-Gwaiz (2008) T is a *Hilbert- Schmidt operator* if there exist an operator basis

$$e_{n=1}^{\infty} : \sum_{n=1}^{\infty} \|Te_n\|^2 < \infty. \tag{1.4}$$

Vectors which have complex components are symbolized by $\mid a\rangle$ and they can also be obtained by linear combination of a set of basis vectors i.e.,

$$\mid a\rangle = c_1 \mid x_1\rangle + c_2 \mid x_2 ... c_n \mid x_n\rangle = \sum_j c_j x_j \tag{1.5}$$

where c_j are constant coefficients (Okelo, 2015). If position vector R is in three dimensions, each vector space can be represented by basis vectors of i, j and k as

$$R = x\hat{i} + y\hat{j} + z\hat{k} = \sum_j c_j x_j \tag{1.6}$$

where, $c_1 = x, c_2 = y, c_3 = z, x_1 = \hat{i}, x_2 = \hat{j}, x_3 = \hat{k}$

1.2 Statement of the Problem

Schrödinger equation was initially derived using the knowledge of electromagnetic prototype of wave function and the Einstein theory of relativity to determine the wave function of an electron. Later, the Schrödinger equation was derived using Newtonian mechanics. It was also derived by extending the wave equation for classical fields to photons and simplified using approximations consistent with generalized non-zero rest mass. Hilbert space operators exhibits some interesting properties like unitization, self adjointness, compactness, norms of linear operation, Hilbert Schmidtization, normal operation, Lebesque and Neumann Integrations as well as the spectrum. An alternative derivation of the Schrödinger equation using these Hilbert space operator techniques comes handy . Consequently the application of the Hilbert space operators in obtaining solutions to the Schrödinger equation that shows the probability position of a particle, momentum and energy has been considered in this study.

1.3 Objectives of the Study

1.3.1 Broad Objective:

The main objective of this study was to formulate the Schrödinger equation using Hilbert space operators approach.

1.3.2 Specific Objectives:

The specific objectives of this study were to:

(i) To establish the correlation between electromagnetic wave theory and Einstein theory of relativity in the derivation of Schrodinger equation using Hilbert space operators.

(ii) To describe the probability position of a particle in three dimensions using Hilbert Space Operators.

(iii) To describe momentum and energy of quantum mechanics using Hilbert Space Operators.

1.4 Significance of the Study

Derivation of Schrödinger equation and its application using Hilbert space operators enhances the understanding of the concept of the Schrödinger equation. The study of Hilbert operators has been useful to mathematical areas such as Approximation theory, Banach Fixed point and the spectral theory as well as the theory of measure and Lebesgue Integration. Approximation theorem is applied because it is concerned with how functions can be approximated with simpler functions and with quantitatively characterising errors introduced. Banach fixed point theorem is useful in this study because the theorem yields existence and uniqueness theorems for differential and integral equation.

Hilbert space operators guarantees the existence of spectral measures i.e. “observables” that correspond to a “state” i.e. an element in Hilbert space. This benefits are extended to understanding of the Schrödinger equation, which is applied in different areas of quantum mechanics.

CHAPTER TWO
LITERATURE REVIEW

2.1 Derivation of Electromagnetic Wave Equation

Electromagnetic wave equation in a vacuum is derived from Maxwell equations, (Dyson, 1990) described as shown in equations (2.1) - (2.4)

$$\vec{\nabla}.\vec{E} = 0, \quad (Gauss'\ law\ of\ electricity) \tag{2.1}$$

$$\vec{\nabla}.\vec{B} = 0. \quad (Gauss'\ law\ of\ magnetism) \tag{2.2}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (Faraday's\ law\ induction) \tag{2.3}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad (Ampere's\ law) \tag{2.4}$$

where,

$\vec{\nabla}$ three-dimensional gradient operator

$\vec{E}$ electric field

$\vec{B}$ magnetic field flux density.

Taking the curl for $\vec{E}$ field propagated along the x direction by Wang (1986) we obtain,

$$\vec{\nabla} \times \vec{E}(x,t)\hat{j} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E(x,t) & 0 \end{vmatrix} = 0 \tag{2.5}$$

Taking the curl of Faraday’s law and substituting Ampere’s law for a charge and current free region Wang (1986) obtained

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}. \tag{2.6}$$

In three dimensions, wave equation, (Wang, 1986) represented it as shown below

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}. \tag{2.7}$$

Remark 1

The derived electromagnetic wave equation in (2.7) helped in obtaining the correlation between Hilbert space operators and Electromagnetic wave equation. This was later used in the derivation of Schrödinger equation.

2.2 Derivation of Einstein Theory of Relativity

The particle freely falling in a gravitational field results in a constant acceleration, that is, the velocity changes at a constant rate (Einstein, 2015). In his work Einstein (2015) noted that it is impossible for an observer to distinguish between an object freely falling in a gravitational field and some other mechanism of uniform acceleration such as a rocket. Since acceleration describes how objects move through space and time, and free fall in gravity and any uniform acceleration were indistinguishable, that gravity's effect on objects may actually be describable by its direct influence on space itself. By this he proved that placing a heavy bowling ball at the center of the trampoline pad, the center of the pad sag downward. Einstein (2015) then assumed the that if the trampoline pad represents space-time, and the bowling ball a gravitating object, then the sagging of the trampoline represents the curvature of space time under the influence of gravity.

Einstein relativistic expressions can be derived starting from the relativity principle and the classical Lorentz's law (Hamdan *et al.*, 2007). The Lorentz's force equation for point charge in both $\vec{E}$ and $\vec{B}$ field is

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \tag{2.8}$$

where,

q charged particle

$\vec{v}$ velocity of the particle

$\vec{E}$ electric field and

$\vec{B}$ magnetic field flux density.

According to Hamdan *et al.*,(2007), the cartesian components of equation (2.8) are

$$F_x = q(E_x + B_z v_y - B_z v_y) \tag{2.9}$$

$$F_y = q(E_y + B_z v_x - B_x v_z) \tag{2.10}$$

$$F_z = q(E_z + B_y v_x - B_x v_y). \tag{2.11}$$

Applying relativity principles on equations (2.9),(2.10) and (2.11) we obtain

$$F'_x = q(E'_x + v'_y B'_z - B'_y v'_z) \tag{2.12}$$

$$F'_y = q(E'_y + v'_x B'_z - B'_x v'_z) \tag{2.13}$$

$$F'_z = q(E'_z + v'_x B'_y - B'_y v'_x). \tag{2.14}$$

According to Hamdan *et al.*(2007) the relativity principle are represented by equations (2.16)-(2.18) where γ scalar factor

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{2.15}$$

$$v'_x = \frac{v_x - u}{1 - v_x \frac{u}{c^2}} \tag{2.16}$$

$$v'_y = \frac{v_y}{\gamma(1 - \frac{u v_x}{c^2})} \tag{2.17}$$

$$v'_z = \frac{v_z}{\gamma(1 - \frac{u v_x}{c^2})} \tag{2.18}$$

In classical physics, a particle with rest mass m_0 with velocity v has a momentum of $p = m_0 v$ and a kinetic energy of $T = \frac{1}{2} m_0 v^2$ and in relativistic physics,

$$p = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}} = mv \tag{2.19}$$

Squaring both side of equation (2.19) Hamdan *et al.*,(2007) obtained

$$P^2 = \gamma^2(m_0^2 v^2) = m^2 v^2 \tag{2.20}$$

The root for the term $\gamma^2(m_0^2 v^2)$ in equation (2.20), Hamdan *et al.*, (2007) obtained,

$$\epsilon = mc^2\sqrt{1 - \frac{v^2}{c^2}} = \gamma m_0 c^2 = mc^2 \tag{2.21}$$

which is the Einstein Equation.

Equation (2.21) is the relativistic energy ϵ, telling us that the change of mass of a particle is accompanied by change in its energy and vice versa. Hamdan *et al.*,(2007) used the classical Cartesian components of Lorentz's law, relativistic velocities, classical momentum and kinetic energy to derive the relativistic energy as shown in equation (2.22).

$$\epsilon^2 = c^2 p^2 + m_0^2 c^4. \tag{2.22}$$

Remark 2

This relativistic energy derived from this literature was used in the derivation of Schrödinger equation using Hilbert space operators.

2.3 Derivation of Schrödinger Equation

Ward and Volkmer (2006) derived the derivation of Schrödinger equation. In their work they used electromagnetic wave equation and Einstein's theory of relativity. They use the same approach as that used by Schrödinger. However they extended the wave equation for classical fields to photons and generalized to non-zero rest mass particles and using approximations consistent with non-relativistic particles. They considered the one dimension equation:

$$\frac{\partial^2 \vec{E}}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0 \tag{2.23}$$

Equation (2.23) is satisfied by plane wave solutions

$$E(x,t) = E_0 e^{i(kx - wt)} \tag{2.24}$$

where $k = \frac{2\pi}{\lambda}$ and $\omega = 2\pi f$ are spatial and temporal frequencies respectively. Substituting equation (2.24) in (2.23) Ward and Volkmer (2006) obtained

$$(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})E_0 e^{i(kx-\omega t)} = 0 \tag{2.25}$$

On solving the wave equation, the dispersion relation for light in free space is $k = \frac{\omega}{c}$ where c is a wave propagation speed. In this case speed of light is in vacuum. From Einstein (2015), the energy of photon is $\epsilon = hv = \hbar\omega$ and the momentum of photon is

$$p = \frac{h}{\lambda} = \hbar k \tag{2.26}$$

Therefore equation (2.24) becomes

$$E(x,t) = E_0 e^{\frac{i}{\hbar}(px-\epsilon t)} \tag{2.27}$$

and on substituting on equation(2.25), Hamdan *et al.*, (2007) obtained

$$-\frac{1}{\hbar^2}(p^2 + \frac{\epsilon^2}{c^2})E_0 e^{\frac{i}{\hbar}(px-\epsilon t)} = 0 \tag{2.28}$$

where, $\epsilon^2 = p^2c^2$.

Since Ward and Volkmer (2006) were dealing with electric field, they replaced E with Ψ, the wave function. Therefore,

$$-\frac{1}{\hbar^2}(p^2 - \frac{\epsilon^2}{c2} + m^2c^4)\Psi_0 e^{\frac{i}{\hbar}(px-\epsilon t)} = 0. \tag{2.29}$$

For the relativistic total energy $\epsilon^2 = p^2c^2 + m^2c^4$ i.e

$$\epsilon = mc^2\sqrt{1 + \frac{p^2}{m^2c^2}}. \tag{2.30}$$

Expanding equation (2.30) binomially, we get

$$\epsilon \approx mc^2(1 + \frac{1}{2}\frac{p^2}{m^2c^2}) = mc^2 + \tau. \tag{2.31}$$

where τ is the classical kinetic energy.

Thus equation $\Psi(x,t) = \Psi_0 e^{\frac{i}{\hbar}(px-mc^2t-\tau t)}$ can be written as

$$\Psi(x,t) = \Psi_0 e^{\frac{i}{\hbar}(px-mc^2t-\tau t)}$$

$$= e^{-\frac{i}{\hbar}mc^2t}\Psi_0 e^{\frac{i}{\hbar}(px-\tau t)} \tag{2.32}$$

Let

$$\Psi_0 e^{\frac{i}{\hbar}(px-\tau t)} = \Phi \tag{2.33}$$

Therefore equation (2.32) can be written as

$$\Psi(x,t) = e^{-\frac{i}{\hbar}mc^2t}\Phi. \tag{2.34}$$

Carrying out second differentive with respect to t on equation (2.34) Ward and Volkmer (2006) obtained,

$$\frac{\partial^2\Psi}{\partial t^2} = \left(-\frac{m^2c^4}{\hbar^2}e^{-\frac{i}{\hbar}mc^2t}\Phi - \frac{2i}{\hbar}mc^2e^{-}\frac{i}{\hbar}mc^2t\frac{\partial\Phi}{\partial t}\right) + e^{-\frac{i}{\hbar}mc^2t}\Phi. \tag{2.35}$$

The first term in brackets is large and the last term is small. We keep the large terms and discard the small one. Using this approximation in the Klein-Gordon equation

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi - \nabla^2\Psi + \frac{m^2c^2}{\hbar^2}\Psi = 0 \tag{2.36}$$

Ward and Volkmer (2006) arrived at the the Shrondinger equation for free particle as shown in equation (2.36)

$$-\frac{\hbar^2}{2m}\nabla^2\Phi = i\hbar\frac{\partial\Phi}{\partial t} \tag{2.37}$$

where Φ is a non-relativistic wave function.

Remark 3

The derivation of Schrödinger equation using electromagnetic wave equation and Einstein theory of relativity, helped in the derivation Schrödinger equation using Hilbert space operators.

2.4 Hilbert Space Operators in Quantum Mechanics

The wave in Hilbert spaces, functions and hence the function of a state are seen as points in the space and each point will have a position vector. The position vector R in 3 dimension is given by

$R = x\hat{i} + y\hat{j} + z\hat{k} = \sum_j c_j x_j$, where $c_1 = x, c_2 = y, c_3 = z$, and $x_1 = \hat{i}, x_2 = \hat{j}, x_3 = \hat{k}$.
Okelo (2015) noted that properties of vector quantities are as follows,
Let vectors which have complex components be symbolized by $\mid a\rangle$, then the conjugate of $\mid a\rangle$ will be given by $\mid a\rangle^\star$

(i) $\langle b \mid c \mid a\rangle = c\langle b \mid a\rangle$

(ii) $\langle b \mid (\mid a\rangle + \mid a'\rangle)\rangle = \langle b \mid a\rangle + \langle b \mid a'\rangle$

(iii) $\langle a \mid a\rangle \geq 0$ and is real

(iv) $\langle a \mid 0\rangle = 0$

(v) If vectors $|a\rangle$ and $|b\rangle$ are orthogonal then $\langle a|b\rangle = 0$

(vi) If $|a\rangle$ and $|b\rangle$ are a unit vectors then$\langle a|b\rangle = 1$

In the study of properties of Hilbert space operators and their application in Schrödinger equation, Okelo (2015) used unconventional vector quantity and properties of Hilbert space to express one- dimensional time dependent wave function $|\Psi(x,t)\rangle$ as a vector ($|\Psi\rangle$) with the total probability that a particle is in a region is represented as,

$$\langle\Psi|\Psi\rangle = \int_{-\infty}^{\infty} \Psi^\star(x,t)\Psi(x,t)dt = 1 \tag{2.38}$$

if $|\Psi\rangle$ is a normalized.

In quantum mechanics, operators are said to be self adjoint if $\hat{A}^\star = \hat{A}$. The adjoint of the momentum operator is $P = P^\star = -i\hbar\frac{\partial}{\partial x}$. The wave function of a free particle is of the form $\Psi(x,t) = Ae^{i(kx-\omega t)}$ so the momentum operator operating on wave function of free particle is given by

$$p \mid \Psi\rangle = -i\hbar(ik)Ae^{i(kx-t)} = \hbar k|\Psi\rangle \tag{2.39}$$

Okelo (2015) noted that the quantum states of physical system are identified as vectors where else the wave functions are viewed as function vectors. This means for one dimensional time dependent wave function $\Psi(x,t)$ can be written as a vector $|\Psi\rangle$. The normalization of wave function is expressed as equation (2.37).

2.4.1 Time Dependent Hilbert Space and Dynamical Invariant

The metric of Hilbert space is not an observable quantity so it does not appear in the Schrödinger equation. The formulation of quantum mechanics on Hilbert space with time dependent inner product arises when developing a non- relativistic quantum mechanics of a particle confined to move in an oscillating membrane of a particle subject to a time dependent in homogeneous gravitation field such as gravitational wave (Mostafazadeh, 2004). In his work, Mostafazadeh (2004) stated that, if $H_1 = H_2$ are Hilbert spaces obtained by endowing a vector space with time independent inner product $\langle .,. \rangle_1$ and time dependent inner product $\langle .,. \rangle_2$ respectively, then a linear operator $\mathcal{H}$ such that $v \to v$ defines a unitary time evolution in H_2 according to Schrödinger equation (1.2)

Koopman (1931) stated that a single particle described by Hamiltonian $H(p,q)$ in two dimensional classical space consists of commuting position and momentum variables and the distributing function is given by $f(q,p;t)$. He showed that the classical Liouville equation obeyed by this distribution is given by

$$\frac{\partial f(q,p;t)}{\partial t} = (\frac{\partial H}{\partial q}\frac{\partial}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial}{\partial q})f(q,p;t) = i\hat{L}f(q,p;t) \tag{2.40}$$

This was immediately followed up by Neumann (1932) who postulated that equation (2.40) can be looked upon as arising from classical square integrable wave functions $\Psi(q,p;t)$ in the Hilbert space of classical phase-space variables $\Psi(q,p;t)$, obeying the time-development equation.

$$\frac{\partial \Psi(q,p;t)}{\partial t} = (\frac{\partial H}{\partial q}\frac{\partial}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial}{\partial q})\Psi(q,p;t) = i\hat{L}\Psi(q,p;t) \tag{2.41}$$

Rajagopal and Ghose (2016) introduced the Hermitian operator $\lambda_{Bi} = -i\partial B_i$. The Liouville equations for the classical wave function and its adjoint (sum on repeated indices assumed) in the new variables

$$i\frac{\partial}{\partial t}\Psi(x,t) = (\frac{\partial}{\partial B_i^\star}\frac{\partial}{\partial \lambda_{Ei}}B_i^\star \lambda_{Ei})B_i^\star)\Psi(x,t) \tag{2.42}$$

Therefore,

$$i\frac{\partial}{\partial t}\Psi^{\star}(x,t) = (\frac{\partial}{\partial B_i^{\star}}\frac{\partial}{\partial \lambda_{Ei}}B_i^{\star}\lambda_{Ei})B_i^{\star})\Psi^{\star}(x,t) \tag{2.43}$$

2.4.2 Relativistic Position of N- Particles

According to Lienert *et al.*,(2017), the non-relativistic quantum mechanics of N particles in three spatial dimensions, the wave function

$$\Psi(q_1, q_2 \ldots, q_N, t) \tag{2.44}$$

is a function of $3N$ position co-ordinates and one time co-ordinate. In relativistic setting, they replaced equation (2.44) by $\Phi(t_1, (q_1, q_2, \ldots, q_N)$). The relation between Ψ and ϕ is reference frame to which Ψ refers to set of all time variables in Φ equal to

$$\Phi(q_1, q_2, \ldots, q_N), t = \Phi((t_1, q_1), (t_2, q_2) \ldots (t_N, q_N)) \tag{2.45}$$

A time evolution law for Φ is a law that determines Φ on its entire domain from initial data. The initial datum is $\Phi(t = 0)$. The kind of evolution analogous to the Schrödinger equation, we set $\hbar = 1$, then

$$i\frac{\partial \Phi}{\partial t} = H\Psi \tag{2.46}$$

is a partial differential equation.

Multi-time wave functions as shown in equation (2.44) arise naturally when considering a particle-position representation of a quantum state in a relativistic setting.

2.4.3 Probability Position of a Particle

For a particle moving in $\mathbb{R}^3$, the wave function Ψ is in Hilbert space $L^2(\mathbb{R}^3)$. The function Ψ corresponds to probability density and is denoted by $\|\Psi\|$. Gagne (2013) stated that the function $\Psi(x, t)$ of position x and time t is such that the probability of a particle being found in a region Δ is given by

$$pr(\Psi, x) \in \Delta = X_{\Delta}\Psi(x,t) = \int dx \|\Psi(x,t)\|^2 = 1 \tag{2.47}$$

and

$$\langle \Psi, X_{\mathbb{R}^3} \rangle = \langle \Psi, \Psi \rangle \tag{2.48}$$

Therefore

$$pr(\Psi, x \in \mathbb{R}^3) = 1 \tag{2.49}$$

In his work Gagne (2013) showed that, for a given projection $P_k : H \to K$, a system is in K if $\Psi \in K$ and $\langle P_K\Psi, \Psi \rangle = 1$,so $\|\Psi\|^2 = 1$. The system is not in K if $\Psi \notin K$ and $\langle P_K\Psi, \Psi \rangle = \langle 0, \Psi \rangle = 0$, where $P_K\Psi, \Psi$ is the probability that a state Ψ lies in K.

A non stationary paticle in a given region posseses energy velocities and momentum. From the theoretical investigations at higher energies, the velocity of Dirac electron is not proportional to momentum operator. To link the relativistic and non-relativistic treatments, Barnett (2017) used Fold-Wouthuysen transformation, $\Psi \to \Psi^{\star} = U\Psi$ where U is a 4×4 matrix as it is in low energy limit of the form that leads to Schrödinger equation. He did this by introducing a unitary operator $e^iS = e^{\beta\alpha.P\frac{\theta}{p}}$ to transform spinor to $\Psi' = e^{iS}\Psi$. The transformation diagnoses the Dirac Hamiltonian

$$H' = e^{iS} = \beta(P^2 + m^2)^{\frac{1}{2}} \tag{2.50}$$

So the transformed Dirac equation is

$$i\frac{\partial \Psi'}{\partial t} = \beta(P^2 + m^2)^{\frac{1}{2}} \tag{2.51}$$

2.5 Properties of Operators in Hilbert Space

Several theorems, lemma and propositions based on properties of operators shall be essential in the sequel especially in obtaining our results as follows;

Theorem 2.5.1: (Remling, 2002)

Let $S, T \in B(\mathcal{H}), c \in \mathbb{C}$, then:

(i) $T^\star \in B(\mathcal{H})$

(ii) $(S+T)^\star = S^\star + T^\star$

(iii) $(cT)^\star = \bar{c}T^\star$

(iv) $(ST)^\star = T^\star S^\star$

(v) $(ST)^\star = T^\star S^\star$

(vi) $(T^\star)^\star = T$

(vii) If T is invertible then $T^\star$ is also invertible and $T^\star T^{(-1)} = T^{(-1)}T^\star$

(viii) $\|T\| = \|T^\star\|, \|TT^\star\| = \|T^\star T\| = T\|^2$.

Theorem 2.5.2: (Remling, 2002)

Let $U \in B(\mathcal{H})$. The following statements are equivalent:

(i) U is unitary

(ii) U is bijective and $\langle Ux, Uy\rangle = \langle x, y\rangle$ for every $x, y \in H$

(iii) U surjective and isometric $\|Ux\| = \|x\|$.

Theorem 2.5.3 (Remling, 2002)

Let $P \in B(\mathcal{H})$. Then the following are equivalent:

(i) P is a projection

(ii) $1 - P$ is a projection

(iii) $P^\star = P$ and a self adjoint

(iv) $P^2 = P$ and P is normal.

Proposition 2.5.4:(The polarization identity) (Kreyszig, 1978)

Let S be a sesquiliear form and Let $q(x) = S(x,x)$

then

$$S(x,y) = \frac{1}{4}[q(x+y) - q(x-y) + iq(x-iy) - iq(x+iy)] \tag{2.52}$$

Theorem 2.5.5 (Projection): (Kreyszig, 1978)

Let M be a closed linear subspace of Hilbert space $\mathcal{H}$. Then every $a \in \mathcal{H}$ can be uniquely written as $a = a_{\|} + a_{\perp}$ with$a_{\|} \in M$and $a_{\perp} \in M^{\perp}$and $H = M \bigoplus M^{\perp}$where $M^{\perp}$is the orthogonal complement of M.

Theorem 2.5.6: (Sunder, 2016)

Let M be a closed subspace of $\mathcal{H}$. Let $e_i : i \in I$ be any orthonormal bais for M and let $e_j : j \in J$ be any orthonormal set such that $e_i : I \cup J$ is orthonormal basis for $\mathcal{H}$. Then the index I and J are disjoint then the following conditions on vector $x \in H$ are equivalent.

$$\begin{aligned} &x \perp y \qquad\qquad \forall y \in M \\ &x = \textstyle\sum_(j \in J), x\langle x, e_j \rangle e_j \end{aligned}$$

Theorem 2.5.7: (Gagne, 2013)

(Parallelogram identity)

let E be a normed space. Then there is an inner product on E which gives rise to the norm if and only if the parallelogram identity $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ is satisfied for all$x, y \in E$

Theorem 2.5.8: (Remling, 2002)

The product of two normal operators is itself normal if and only if the operators commute.

Theorem 2.5.9: (Gagne, 2013)

If T is self adjoint operator on Hilbert space $\mathcal{H}$ then $\|T\| = sup\{| \langle Tx, x \rangle |: \|x\| = 1\}$.

Remark 3

For the sake of further reference, the proves for theorems (2.5.10), (2.5.11) and (2.5.14) are provided.

Theorem 2.5.10: Remling (2002)

If T is idempotent self adjoint operator then T is a projection of $M = \{x \in H : Tx = x\}$

Proof. Let $Z \in \mathcal{H}$ and write it as $Z = TZ + (Z - TZ)$

$$T(TZ = TZ) \text{ so } TZ \in M$$

$$\text{and } Z - TZ \in M^{\perp}.$$

$$\text{If } x \in M, \text{ then}$$

$$\langle x, Z - TZ \rangle = \langle x, Z \rangle - \langle x, TZ \rangle = \langle x, Z \rangle - \langle Tx, Z \rangle = 0 \tag{2.53}$$

□

Theorem 2.5.11: Akhiezer and Glazman (2013)

If P is a nonzero orthogonal projection, then $\|P\| = 1$.

Proof. If $x \in H$ and $Px \neq 0$, then the use of the Cauchy-Schwarz inequality implies that

$$\|P\| = \frac{\langle Px, Px \rangle}{\|Px\|} = \frac{\langle x, P^2x \rangle}{\|Px\|}$$

$$= \frac{\langle x, Px \rangle}{\|Px\|} \leq \|x\| \tag{2.54}$$

□

if $P \neq 0$, then there is an $x \in \mathcal{H}$ with $Px \neq 0$ and $\|P(Px)\| = \|Px\|$ thus $\|P\| \geq 1$.

Proposition 2.5.12: Degli Esposti et al. (2006)

Suppose that T is a bounded linear operator on a separable Hilbert space H such that there is an orthonormal, then

$$e^{\infty}_{(n=1)} : \sum\nolimits^{\infty}_{(n=1)} \langle Te_n \rangle^2 \leq \infty$$

for any orthogonormal basis $f_{n(n=1)}^{\infty}$ then,

$$\sum_{(n=1)}^{\infty} \|Tf_n\rangle^2 = \sum_{(n=1)}^{\infty} \|Te_n\|^2$$

Theorem 2.5.13: Conrad (1998)

(Fixed Point Theorem)

Let (X, d) be a complete metric space and $f \to X$ be a map such that $d(f(x), f(x)) \leq cd(, x)$ for some $0 \leq c \leq 1$ and for all $x, x \in X$. Then f has a unique fixed point in X. Moreover, for any $x_0 \in X$. The sequence iterates $x_0, f(x_0), f(f(x_0))$ converges to the fixed point of $f(x)$. Where $d(f(x), f(x)) \leq cd(x, x)$. Then $f(x)$ is called contraction.

Theorem 2.5.14: Putnam (2012)

If B is any bounded operator and if A is normal and not necessarily bounded and if $BA \subset AB$ then $BA^{\star} \subset A^{\star}B$

Proof. From disjoint Borel sets of complex plane given as

$$Q = K(\alpha_1)BK(\alpha_2) = 0$$

$K(\alpha_1)$ denotes the projection operator with Borel set α by spectral family K_z.

Suppose α_1 and α_2 are bounded then

$$B\int_{(}\alpha_2)ZdK_zx = ABK(\alpha_2)x$$

Applying the operator $K(\alpha_1)$ Then

$$K(\alpha_1)B\int_{(\alpha_2)} ZdK_z = \int_{(}\alpha_1)ZdK_zBK\alpha_2$$

If z_1 and z_2 are arbitrary numbers in α_1 and α_2 respectively, then the above equation can be written as

$$\int_{(\alpha_1)}(Z - Z_1)dK_zQ = Q\int_{(\alpha_2)}(Z - Z_2)dK_z + (Z_2 - Z_1)Q$$

Let α denote any Borel set then

$$K(\alpha)B = K(\alpha)BI = K(\alpha)B(K(\alpha)) + K(\alpha') = K(\alpha)BK(\alpha)$$

α' is the complement of α. Similary,

$$BK(\alpha) = K(\alpha)B(K(\alpha))$$

Therefore,

$$K(\alpha)B = BK(\alpha).$$

This implies that

$$BA^{\star} \subset A^{\star}B$$

$\square$

Remark 4

The above properties and theorems of Hilbert space shall be used in the derivation and study of applications of Schrödinger equation using Hilbert space approach. The study shall also establish the correlation of the abstract mathematical application of the Hilbert space operators in relation to the application of Schrödinger equation in quantum mechanics.

CHAPTER THREE
METHODOLOGY

3.1 General Approach

For the successful completion of this study, background of quantum mechanics, Hilbert space operators and their application on Quantum mechanics is required.

3.2 Technical Approach

Specifically, for effective completion of this work the following approaches were employed. The properties of Hilbert space operators, Reimann's intergral compact operators, norms of linear operators, Hilbert Schmidt operator, normal operators, Neumann Integral Lebesque integral, Reimann's integral and spectrum.

3.2.1 Hilbert Space Operators

In this section, we would like to review known results on Hilbert space operators which include properties and theorems that were very instrumental in obtaing our results.
Young (1988) stated that, Hilbert space is a complete inner product with the following properties.

(i) Orthogonally property $\langle x, y\rangle = 0$

(ii) Self- adjointness $\langle x, y\rangle = \overline{\langle y, x\rangle}$

(iii) Unitary $\langle x, x^\star\rangle = 1$ if and only if $x = x^\star$

(iv) Positivity $\langle x, x\rangle \geq 0 = \|x\|^2$

(v) Nullity property $\langle x, x\rangle = 0$ if and only if $x = 0$.

From section 1.1 the properties of an operator on Hilbert space are as follows. If T is an operator on Hilbert space $\mathcal{H}$ then:

(i) T is normal if $TT^\star = T^\star T$

(ii) T is self-adjoint (or Hermitian) if $T = T^\star$

(iii) T is positive if $\langle Tx, x\rangle \geq 0$ for all $x \in H$.

(iv) T is unitary if $TT^\star = T^\star T = 1$.

These properties are applied in establishing the electromagnetic wave equation and Einstein theory of relativity in correlation with Hilbert space operator as well as in the alternative derivation of Schrödinger equation. In addition, these properties are useful in establishing the application of Hilbert space operators in quantum observable.
Theorem (3.2.1) is useful in establishing the probability position of a particle.

Theorem 3.2.1 (Spectral Theorem)

Let A be a compact self-adjoint operator on a Hilbert space H with a complete orthonormal system of eigenvectors $v_1, v_2, v_3 \dots$ with corresponding eigenvalues $_1, _2, _3 \dots$. Let $P_i : \mathcal{H} \rightarrow \mathcal{H}$ be the one dimensional projection onto span $v_1, v_2, v_3 \dots$ mapped by $x \rightarrow x, v_i$, then for all $x \in \mathcal{H}$ can be written as

$$x = \sum_{i=1}^{\infty} P_i x$$

$$A = \sum_{i=1}^{\infty} P_i x$$

From the existing literature, orthonormal system $B' = u_1, u_2, u_3, \dots$ forms a basis for vectors in vector space S. Therefore if we add an arbitrary orthonormal basis for $S^\perp$ to the set B' we obtain a complete orthonormal system $B\prime = v_1, v_2, v_3, \dots$. Given a complete orthonormal basis B we may write any $x \in H$ as

$$x = \sum_{i=1}^{\infty} \langle x, v_i \rangle v_i.$$

With our definition of P_i as $P_i x = \langle x, v_i \rangle$ for all $i \in \mathbb{N}$ we have

$$x = \sum_{i=1}^{\infty} P_i x$$

and since our basis consist of eigenvectors we also have

$$Ax = \sum_{i=1}^{\infty} \lambda_i \langle x, v_i \rangle v_i = \sum_(i = 1)^{\infty} \lambda_i P_i x.$$

This results to

$$A = \sum_{i=1}^{\infty} \lambda_i P_i.$$

Suppose $\mathcal{H}$ is a separable Hilbert space and $T \in B(\mathcal{H})$. Then T is a Hilbert- Schmidt operator if there exist an operator basis $e_n^\infty : \sum_{n=1}^{\infty} \|Te_n\|^2 < \infty$ (Degli Esposti *et al* 2006)

3.2.2 Integrals

(a) Reimann's Integrals

Reimann's integrals are useful in the application of Hilbert space operators in quantum observables such as in describing the determination of position of a particle in three dimensions. A fuction f is said to be Riemann integrable on $[a, b]$ if there is a number L with the following property. For every $\epsilon > 0$ there exists a $\delta > 0$ such that $|\sigma - L| < \epsilon$. If σ is Riemann's sum of f over partition P of $[a, b]$ such that $\|P\| < \delta$. Then L is Riemann's integral of f over

$$\int_a^b f(x)dx = L$$

According to Karatsuba and Voronin (2011), the following are properties of Reimann's integral:

(i) Let f be a Riemann integrable function on $[a, b]$ If $f \geq 0$ on $[a, b]$, then

$$\int_a^b f(x) \geq 0.$$

If f is continuous on $[a, b]$ and $f > 0$ on $[a, b]$, then

$$\int_a^b f(x) > 0.$$

(ii) Let f be a Riemann's integrable function on $[-a, a]$. If f is an odd function, then

$$\int_{-a}^{a} f(x) = 0. \tag{3.1}$$

If f is an even function, then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx. \tag{3.2}$$

(iii) Let f be a Riemann integrable function on $[a, b]$. If m, M are real numbers such that $m \leq f \leq M$ on $[a, b]$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a). \tag{3.3}$$

(iv) Let f and g be Riemann integrable functions on $[a, b]$. If $f \leq g$ on $[a, b]$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx. \tag{3.4}$$

(v) Let f be a Riemann integrable function on $[a, b]$. Then its absolute value $|f|$ is a Riemann integrable function on $[a, b]$ and

$$\int_a^b |f(x)dx| = \int_a^b \mid f(x) \mid dx. \tag{3.5}$$

b) Neumanns Integral

Neumann's integral follows from Fredholm integral. Landweber (1951) stated that, given that the Fredholm integral equation is in the form

$$f(x) = \lambda \int_a^b k(x, y)f(y)dy + g(x) \tag{3.6}$$

where k is the continuous kernel function and g and f are functions.

A function is said to be Neumanns integrable if it is in the form,

$$f_0(x) = g(x) \qquad\qquad \lambda^0$$

$$\int f(x) = \int_a^b k(x, y) + f_{n-1}(y)dy \qquad \lambda^n; n > 1 \tag{3.7}$$

where λ is the eigen value.

c) Lebesque Integral

The properties of Lebesque integral has been used in this study in the determination of probability position of a particle. For a Lebesque integral, Burkill (2004) stated that Suppose that $f : \mathbb{R} \rightarrow \mathbb{C}$ is a non-negative real-valued function, then f is Lebesque integrable if

$\int f d\mu = \int_0^\infty f^\star(t)dt.$

Using the "partitioning the range of f " philosophy, the integral of f should be the sum over t of the elementary area contained in the thin horizontal strip.

The knowledge on the space L^2_σ was of great use in this study. By Carter and Van Brunt (2000), let a non degreasing function of a bounded variation $\sigma(t) - \infty < t < \sigma)$ be given then it is left countinous : $\sigma(t-0) = \sigma(t)$. Such fuctions are refered to as distribution function. From this distribution function, it is possible to construct a measure analogous to lebesque measure of interval $[a, b]$ for $a \leq b$. This can be replaced by σ - length $\sigma(b+0) - \sigma(a)$. This means, some intevals may have σ- length different from zero and some proper intervals may have σ-length equal to zero. The measure determined by this σ- length is called σ-*measure* and is a measurable function corresponds to Lebesque- Stieltjes integral.

We consider a linear space σ –measurable functions f for which Lebesque- Stieltjes integral is

$$\int_{-\infty}^{\infty} \mid f(t) \mid^2 d\sigma(t). \tag{3.8}$$

Metrizing it by means of metric generated by scalar product then,

$$(f, g) = \int_{-\infty}^{\infty} f(t)g(\bar{t})\sigma d(t). \tag{3.9}$$

This linear space is complete and therefore is a Hilbert space denoted by L^2_σ.

3.2.3 Properties of Einstein Theory of Relativity

To obtain the correlation of Hilbert space operators with Einstein theory of relativity, we used properties of Hilbert space operators, derivations and properties of Einstein theory of relativity. Levy-Leblond (1976) derived the special relativity. He stated that, if the observer O' in a system S' looks at a ray of light passing through the origin and observes that it satisfies the equation.

$$x' = ct' \tag{3.10}$$

where x is the displacement and c is the velocity of light

An observer O in a system S looking at the same ray, he observes that it satisfies the equation

$$Ax + Bct = Gx + Dct$$

$$(A - G)x = (D - B)ct \tag{3.11}$$

where A, B, G and D are constants.

Since the velocity of light is c for all observers then

$$A - G = D - B \tag{3.12}$$

Therefore for observer O

$$x = ct. \tag{3.13}$$

If observer O' is looking at a ray going opposite direction then equation (3.13) becomes

$$x' = -ct'. \tag{3.14}$$

To observer O, this ray satisfies

$$Ax + Bct = -Gx - Dct \tag{3.15}$$

$$(A + G)x = (D + B)ct. \tag{3.16}$$

Since the velocity of light is c for all observers then

$$(A + G) = (D + B). \tag{3.17}$$

Adding equation (3.12) and (3.17) we obtain $A = D$ and $B = G$.

Thus if observers O and O' can agree that their systems are moving past each other with velocity v, we can now write the transformation equation as

$$x' = Ax - Avt \tag{3.18}$$

$$ct' = -Av\frac{x}{c} + Act. \tag{3.19}$$

We suppose that there is a measuring rod of length L, fixed with respect to S, and with one end at the origin. Also suppose there is another rod of length L with one end at the origin of S', then the apparent lengths of the rods must be the same. Thus according to the principle of relativity, there should be no difference between the observations of O and O'.

$$\frac{L}{A} = LA(1 - \frac{v^2}{c^2}) \tag{3.20}$$

$$A = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{3.21}$$

Substituting equation (3.20) in (3.18) and (3.19) respectively we obtain the Lorentz transformation as shown in equation (3.22) and (3.23)

$$x' = \frac{(x - vt)}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{3.22}$$

$$ct' = \frac{(ct - vt)}{1 - \frac{v^2}{c^2}}. \tag{3.23}$$

Equation (3.22) and (3.23) are applicable in establishing the correlation of Einstein theory of relativity with Hilbert space operators.

3.2.4 Electromagnetic Wave Equation

By the use of properties of Hibert space operators, Maxwell differential equations, and the existing derivation of electromagnetic wave equation, we were able to obtain the the correlation of electromagnetic wave equation and Hilbert space operators. The properties of Hilbert space operators follows from section (3.2.1), the Maxwell differential equations and the existing derivation of electromagnetic wave equation follows from section (2.1).

3.2.5 Quantum Observables

Quantum observables include, position, momentum and energy. This study has focused on describing probability position of a particle in three dimension, momentum and energy which particles possess. By Heslot (1985), Let g be an observable, and $\hat{g}$ the corresponding self-adjoint operator the value of g when the system is in the state described by the normalized vector $\mid \Psi\rangle$ is given by $\langle \Psi \mid \hat{g} \mid \Psi\rangle$.

To describe the probability position of a particle in three dimension using Hilbert space operators, literature on probability position of a particle in section 2.4.3 is used together with Hilbert space operators. To describe the momentum of a particle, we used the concept on how properties of Hilbert space operators correlate with momentum as well as the Fourier transform of $\Psi(x)$ in momentum representation.

El Naschie (2013) stated that

$$\langle P \mid \Psi\rangle = \int dx \langle P \mid x\rangle\langle x \mid \Psi\rangle. \tag{3.24}$$

Inserting a complete set of momentum states we obtain

$$\langle x \mid \hat{P} \mid \Psi\rangle = \int dP\langle x \mid P\rangle\langle P \mid \hat{P} \mid \Psi\rangle = \int dpPx \mid P\rangle\langle P \mid \Psi\rangle. \tag{3.25}$$

From the equation (3.25) we obtain

$$\langle P \mid x \mid P\rangle = \int \frac{\hbar}{i}\langle x \mid \psi\rangle. \tag{3.26}$$

Therefore, equation (3.29) becomes

$$\langle x \mid \hat{P} \mid \Psi\rangle = \int dp\frac{\hbar}{i}\frac{\partial}{\partial x}\langle x \mid P\rangle\langle P \mid \psi\rangle \tag{3.27}$$

$$\langle x \mid \hat{P}\Psi \mid\rangle = \int dp\frac{\hbar}{i}\frac{\partial}{\partial x}\langle x \mid P\rangle. \tag{3.28}$$

The operator for the linear momentum in the x direction is

$$P_x = \frac{\hbar}{i}\frac{\partial}{\partial x}, \tag{3.29}$$

in the y direction

$$P_y = \frac{\hbar}{i}\frac{\partial}{\partial y} \tag{3.30}$$

and in the z direction

$$P_z = \frac{\hbar}{i}\frac{\partial}{\partial z}. \tag{3.31}$$

Therefore the momentum in three dimensions is given as

$$\frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{P_z^2}{2m} = \frac{\hbar}{2m}(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}). \tag{3.32}$$

3.3 Ethical Consideration

The Proposal was cleared and authorization given by Chuka University Ethical Review Committee (Appendix A). The research permit was obtained from National Commission for Science, Technology and Innovation (NACOSTI) before commencing the research (Appendix B). The ethical issues such as plagiarism, falsifying of the work and biasedness were strictly observed. The outcomes of the study were carefully and critically reviewed so that the results were credible and reported with honesty and integrity. The findings will be published in a transparent manner in order to help further knowledge advancement in areas of Pure Mathematics. Finally, the results from this study upon publishing will not be retrieved or transmitted without the knowledge of Chuka University.

CHAPTER FOUR
THE SCHRÖDINGER EQUATION AND OPERATORS IN HILBERT SPACE

4.1 Electromagnetic Wave Theory and Einstein Theory of Relativity in Correlation with Operators in Hilbert Space

This section deals with the correlation of operators in Hilbert space with electromagnetic wave equation. The solutions are obtained from the properties of operators in Hilbert space as shown in section (3.2.1) as well as the existing derivation of electromagnetic wave equation from the existing literature in section 2.1.

We have also described the correlation of Einstein theory of relativity with Operators in Hilbert space using the existing literature in section 2.2 and properties of Operators in Hilbert space as listed in section 3.2.1.

4.1.1 Correlation between Operators in Hilbert Space and Electromagnetic Wave Theory

Hilbert Space is a complete inner product space. Dirac invented an alternative for inner product that leads to bras $\langle . \mid$ and kets $\mid . \mid$ (Roberts, 1966). That is,

$$\langle x, y \rangle \to \langle x \mid y \rangle.$$

Bra-kets have the following properties

(i) $\langle x \mid y \rangle = 0$ if both x and y are orthogonal

(ii) $\langle x \mid x \rangle = 0$ iff $x = 0$ (null property)

(iii) $\langle x \mid x \rangle \geq 0 = \|x\|^2$

(iv) $\langle x \mid ay + bz \rangle = a\langle x \mid y \rangle + b\langle x \mid z \rangle$

Properties of dot product are similar to that of inner product. They include:

(i) $x.x = |x|^2$

(ii) $x.y = y.x$

(iii) $a.(b + c) = a.b + a.c$

(iv) $ea.b = e(a.b) = a(eb)$ for e is a scalar and a, b, c are vectors

Electromagnetic waves are electric and magnetic waves that travel perpendicular to each other. Young (1988) showed that, these waves are orthogonal and can be represented as

$$\langle E, B \rangle = 0.$$

They have Amplitude, Wavelength and Frequency.

Electromagnetic wave equation is a second order partial differential equation which describes electromagnetic waves through a medium or a vacuum. The vector differential operator is given as

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}. \tag{4.1}$$

Maxwell equations describe the world of electromagnetic, that is, how electric and magnetic field interact. Applying the properties of inner product on Maxwell equations, they can be represented as follows;

$$\langle \vec{\nabla}, \vec{E} \rangle = 0 \ \ (Gauss'\ law\ of\ electricity) \tag{4.2}$$

$$\langle \vec{\nabla}, \vec{B} \rangle = 0 \ \ (Gauss\ law\ of\ magnetism) \tag{4.3}$$

$$\vec{\nabla} \times \vec{E} = \frac{\partial \vec{B}}{\partial t} \ \ (Faraday\text{'}s\ law\ induction) \tag{4.4}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \ \ (Ampere\text{'}s\ law). \tag{4.5}$$

In the derivation of Schrodinger equation, procedure used by Wang (1986)is used but applying properties of Hilbert space operators. For non-conducting media, or in a vacuum, there are no sources and hence, $\rho = 0$,and $\sigma = 0$ Where μ and ε are permeability and permittivity of free space respectively. Since $\vec{\nabla}$ and $\vec{E}$ are both vectors, the Maxwell equation (4.2) can be written as;

$$\langle\vec{\nabla}, \vec{E}\rangle = 0. \tag{4.6}$$

Taking the curl of Faraday's law (equation 4.4) becomes,

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial(\vec{\nabla} \times \vec{B}}{\partial E}). \tag{4.7}$$

Considering the left hand side of equation (4.7) we have

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}, (\vec{\nabla}, \vec{E} - \vec{E}(\vec{\nabla}, \vec{\nabla})). \tag{4.8}$$

From property (iii) of inner product equation (4.8) becomes

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \langle\vec{\nabla}, \langle\vec{\nabla}, \vec{E}\rangle\rangle - \langle\langle\vec{\nabla}, \vec{\nabla}\rangle, \vec{E}\rangle. \tag{4.9}$$

By the first of Maxwell equation, $\langle\langle\vec{\nabla}, \vec{E}\rangle = 0$ in vacuum. Therefore,

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\langle\langle\vec{\nabla}, \vec{\nabla}\rangle, \vec{E}\rangle. \tag{4.10}$$

Consider right hand side of equation (4.7), $\frac{\partial(\vec{\nabla} \times \vec{B})}{\partial E}$ substituting the Ampere's law for a charge and current-free region we have

$$\frac{\partial(\vec{\nabla} \times \vec{B})}{\partial t} = \frac{\partial}{\partial t}\mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}, \tag{4.11}$$

hence the equation we obtain is

$$\langle\langle\vec{\nabla}, \vec{\nabla}\rangle, \vec{E}\rangle = -\frac{1}{\langle c, c\rangle}\frac{\partial^2 \vec{E}}{\partial t^2}. \tag{4.12}$$

We find that each component of the electric field satisfies equation (4.12) which is the derived wave equation using properties of inner product. The quantity c is defined as the speed of the wave and $\mu_0\epsilon_0 = \frac{1}{\langle c, c\rangle}$.

4.1.2 Correlation between Hilbert Space Operators and Einstein Theory of Relativity

Einstein theory of relativity can be derived starting from the relativity principle and the classical Lorentz's law (Hamdan *et al.* 2007) as shown in equation (4.13)

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \tag{4.13}$$

Where, q charged particle

$\vec{v}$ velocity of the particle

$\vec{E}$ electric field and

$\vec{B}$ magnetic field flux density

In this work, Einstein theory of relativity is derived following the procedure followed by Hamdan *et al.* (2007) but using properties of Hilbert space operator.

Since q is a scalar quantity and $\vec{F}$, $\vec{B}$ and $\vec{v}$ are vectors quantities, applying properties of inner product, the Cartesian components of equation (4.13) are given by

$$F_x = qE_x + q\langle v_y, B_z\rangle - q\langle v_z, B_y\rangle \tag{4.14}$$

$$F_y = qE_y + q\langle v_z, B_x\rangle - q\langle v_x, B_z\rangle \tag{4.15}$$

$$F_z = qE_z + q\langle v_x, B_y\rangle - q\langle v_y, B_x.\rangle \tag{4.16}$$

Applying relativity principles on equations (4,14),(4.15) and (4.16) we obtain

$$F'_x = qE'_x + q\langle v'_y, B'_z\rangle - q\langle v'_z, B'_y\rangle \tag{4.17}$$

$$F'_y = qE'_y + q\langle v'_z, B'_x\rangle - q\langle v'_x, B'_z\rangle \tag{4.18}$$

$$F'_z = qE'_z + q\langle v'_x, B'_y\rangle - q\langle v'_y, B'_x\rangle. \tag{4.19}$$

In the derivation of relativistic energy, the three-vector relativistic velocity transformation is necessary. According to Hamdan *et al*., (2007), the relativistic velocity equations but applying the properties of inner product can be written as

$$v'_x = \frac{v_x - u}{1 - \frac{\langle v_x, u\rangle}{\langle c, c\rangle}} \tag{4.20}$$

$$v'_y = \frac{v_y}{\gamma\left(1 - \frac{\langle v_x, u\rangle}{\langle c, c\rangle}\right)} \tag{4.21}$$

$$v'_z = \frac{v_z}{\gamma\left(1 - \frac{\langle v_x, u\rangle}{\langle c, c\rangle}\right)}. \tag{4.22}$$

where scalar factor γ is fixed by applying the relativity principle $\gamma = \frac{1}{\sqrt{1 - \frac{\langle u, u\rangle}{\langle c, c\rangle}}}$

In classical physics, a particle with rest mass m_0 with velocity v has a momentum of $p = m_0 v$ and a kinetic energy of $T = \frac{1}{2} m_0 \langle v, v\rangle$ and in relativistic physics, $p = \frac{m_0 v}{1 - \frac{\langle v, v\rangle}{\langle c, c\rangle}} = \gamma m_0 v = mv$ If we consider two inertial systems S and S', then charged particle q when viewed from S the components of momentum are given by the following as stated by Vecchiato (2017),

$$p_x = m v_x \tag{4.23}$$

$$p_y = m v_y \tag{4.24}$$

$$p_z = m v_z. \tag{4.25}$$

When viewed from S', the momentum is given by

$$p'_x = m' v_x{}' \tag{4.26}$$

$$p'_y = m' v_y{}' \tag{4.27}$$

$$p'_z = m' v_z{}'. \tag{4.28}$$

From (4.23),

$$v_x = \frac{p_x}{m} \tag{4.29}$$

From (4.26)

$$v'_x = \frac{p'_x}{m'} \tag{4.30}$$

Equating (4.30) and equation (4.20) we obtain

$$\frac{p'_x}{m'} = \frac{v_x - u}{1 - \frac{\langle v_x, u\rangle}{\langle c, c\rangle}}. \tag{4.31}$$

Substituting equation (4.29) in (4.31) we obtain

$$\frac{p'_x}{m'} = \frac{p_x - mu}{m(1 - \frac{\langle u, v_x\rangle}{\langle c, c\rangle})}. \tag{4.32}$$

Observers of frame S measures the rest mass m_0, observers from S' measure the mass m'. Assuming the charged particle is at rest then

$$v_x = u = 0. \tag{4.33}$$

Observers of frame S' measures the rest mass m_0, observers from S measure the mass m. Assuming the charged particle is at rest then the component of momentum if combine (4.21), (4.24) and (4.27) we deduce

$$p'_y = m'v'_y = mv_y = p_y \tag{4.34}$$

$$p'_z = m'v'_z = mv_z = p_z. \tag{4.35}$$

The relativistic mass in both frames is expressed as

$$m = \frac{m_0}{\sqrt{1 - \frac{\langle v, v\rangle}{\langle c, c\rangle}}} \tag{4.36}$$

$$m' = \frac{m_0}{\sqrt{1 - \frac{\langle v, v\rangle}{\langle c, c\rangle}}}. \tag{4.37}$$

Multiplying the equation for scalar factor γ by $m_0^2\langle c, c\rangle\langle c, c\rangle$ we obtain

$$\sqrt{1-\frac{\langle v,v\rangle}{\langle c,c\rangle}}(m_0^2\langle c,c\rangle\langle c,c\rangle) = m^2\langle c,c\rangle\langle c,c\rangle \tag{4.38}$$

$$\langle c,c\rangle\langle c,c\rangle\gamma^2 m_0^2 - \langle c,c\rangle\gamma^2 m_0^2\langle v,v\rangle = m^2\langle c,c\rangle\langle c,c\rangle \tag{4.39}$$

$$\gamma^2 m_0^2\langle c,c\rangle\langle c,c\rangle - \gamma^2 m_0^2\langle v,v\rangle, \langle c,c\rangle = m^2\langle c,c\rangle\langle c,c\rangle. \tag{4.40}$$

It is noted that

$$\langle p,p\rangle = \gamma^2(m_0^2\langle u,u\rangle) = m^2\langle v,v\rangle. \tag{4.41}$$

The root for the first term in equation (4.40) is obtained as,

$$\epsilon = m\langle c,c\rangle\sqrt{1-\frac{\langle v,v\rangle}{\langle c,c\rangle}} = \gamma m_0\langle c,c\rangle = m\langle c,c\rangle. \tag{4.42}$$

Therefore,

$$\epsilon^2 = \langle c,c\rangle\langle p,p\rangle + m_0^2\langle c,c\rangle\langle c,c\rangle. \tag{4.43}$$

This is the derived equation for relativistic energy which is used in the derivation Schrödinger equation using of Hilbert space operator.

4.2 The Derivation of Schrödinger Equation using Operators in Hilbert Space

The results obtained in section 4.1.1 and 4.1.2 are utilized in the derivation of Schrödinger equation using Hilbert space approach.

Schrödinger equation was first derived by Erwin in 1926. Ward and Volkmer (2006) derived the Schrödinger equation using electromagnetic wave theory and Einstein theory of relativity as used by Erwin, however he extended the wave equation for classical fields to photons and generalized to non-zero rest mass particles and using approximations consistent with non-relativistic particles. In this study, we used the same approach as used by Ward and Volkmer but applied the properties of Operators in Hilbert space.

Equation (4.12) obtained from derivation of electromagnetic wave equation can be written as,

$$\langle\langle\nabla,\nabla\rangle,\vec{E}\rangle-\frac{1}{\langle c,c\rangle}\frac{\partial^2\vec{E}}{\partial t^2}=0. \tag{4.44}$$

This satisfies

$$E(t,x)=E_0e^{i(kx-\omega t)} \tag{4.45}$$

where $k=\frac{2\pi}{\lambda}$ and $\omega=2\pi f$ are spatial and temporal frequencies respectively. Substituting equation (4.45) in (4.44) we obtained

$$\left(\langle\nabla_x,\nabla_x\rangle,E_0\rangle-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E_0e^{i(kx-\omega t)}=0. \tag{4.46}$$

In a vacuum, the speed of light is given as $c=\lambda$, a wave propagation speed and $k=\frac{\omega}{c}$. From Einstein and Compton, the energy of photon is $\epsilon=hv=\hbar\omega$ and the momentum of photon is

$$p=\frac{h}{\lambda}=\hbar k. \tag{4.47}$$

Therefore equation (4.45) is written as

$$E(x,t)=E_0e^{\frac{i}{\hbar}(px-\epsilon t)}. \tag{4.48}$$

Substituting equation (4.48) in (4.44) we obtain,

$$\left(\langle\langle\nabla_x,\nabla_x\rangle,E_0\rangle-\frac{1}{\langle c,c\rangle}\frac{\partial^2}{\partial t^2}\right)E_0e^{(\frac{i}{\hbar}(px-\epsilon t))}=0. \tag{4.49}$$

on differentiating equation 4.49 we obtain

$$-\frac{1}{h^2}\left(\langle\langle p,p\rangle,E_0\rangle+\epsilon^2\frac{1}{\langle c,c\rangle},\vec{E_0}\rangle\right)e^{\frac{i}{\hbar}(px-\epsilon t)}=0. \tag{4.50}$$

Since $E,\Psi\in S$, where S is a vector space, then replacing electric field, E with Ψ, the wave function equation (4.45) in term of wave function can be written as,

$$\Psi(x,t)=\Psi_0e^{\frac{i}{\hbar}(px-\epsilon t)}. \tag{4.51}$$

Replacing E with Psi equation (4.50) becomes

$$-\frac{1}{h^2}\left(\langle\langle p,p\rangle,\Psi_0\rangle+\epsilon^2\frac{1}{\langle c,c\rangle},\vec{\Psi_0}\rangle\right)e^{\frac{i}{\hbar}(px-\epsilon t)}=0. \tag{4.52}$$

Now, the relativistic total energy obtained from the results in section 4.2.2 is given as

$$\epsilon^2=\langle p,p\rangle\langle c,c\rangle+m^2\langle c,c\rangle\langle p,p\rangle \tag{4.53}$$

therefore,

$$\epsilon=m\langle c,c\rangle\sqrt{1+\tfrac{\langle p,p\rangle}{m^2\langle c,c\rangle}}$$

$$\epsilon\approx m\langle c,c\rangle(1+\frac{1}{2}\frac{\langle p,p\rangle}{2m^2\langle c,c\rangle})=m\langle c,c\rangle+\tau \tag{4.54}$$

where τ is the classical kinetic energy.

Thus equation (4.51) becomes

$$\Psi(x,t)=\Psi_0e^{\frac{i}{\hbar}(px-mc^2t-\tau t)}$$

$$=e^{-\frac{i}{\hbar}mc^2t}\Psi_0e^{\frac{i}{\hbar}(px-\tau t)}. \tag{4.55}$$

Taking $\Psi_0e^{\frac{i}{\hbar}(px-\tau t)}=\Phi$ then it follows,

$$\Psi(x,t)=e^{-\frac{i}{\hbar}mc^2t}\Phi. \tag{4.56}$$

On differentiating equation (4.56) with respect to time (t) we obtain

$$\frac{\partial\Psi}{\partial t}=-\frac{m}{\hbar}\langle\langle c,c\rangle,\Phi\rangle e^{-\frac{i}{\hbar}mc^2t}+e^{(-\frac{i}{\hbar}mc^2t)}\frac{\partial\Phi}{\partial t}. \tag{4.57}$$

Carrying out the second derivative of equation (4.57) we have

$$\frac{\partial^2\Psi}{\partial t^2}=\left(-\frac{m^2\langle c,c\rangle\langle c,c\rangle}{\hbar^2}e^{-\frac{i}{\hbar}mc^2t}\Phi-\frac{2i}{\hbar}m\langle c,c\rangle\frac{\partial\Phi}{\partial t}\right)+e^{(-\frac{i}{\hbar}mc^2t)}\Phi. \tag{4.58}$$

The term $e^{-\frac{i}{\hbar}mc^2t}\frac{\partial\Phi}{\partial t}$ is very small and therefore it can be discarded. The term in brackets is very large thus, using this approximation in the Klein-Gordon equation we obtain

$$e^{-\frac{i}{\hbar}mc^2t}[\langle\nabla^2\Phi^2,\Phi\rangle+\frac{2im}{\hbar}\frac{\partial\Phi}{\partial t}]=0 \tag{4.59}$$

$$\langle \nabla^2, \Phi \rangle + \frac{2im}{\hbar}\frac{\partial}{\partial t} = 0 \tag{4.60}$$

$$-\frac{\hbar^2}{2m}\langle\langle \nabla, \nabla \rangle, \Phi \rangle = i\hbar\frac{\partial}{\partial t}\Phi \tag{4.61}$$

where Φ is the non-relativistic wave function.

Therefore equation (4.61) is the derived Schrödinger equation for free particle.

4.3 Application of Properties of Hilbert Space Operators in Quantum Observables

Mostafazadeh (2004) investigated effects of allowing the Hilbert space of a quantum system to have a time-dependent metric. He stated that, for a given possibly non-stationary quantum system, the requirement of having a unitary Schrödinger time-evolution identifies the metric with a positive-definite dynamical invariant of the system. Therefore the geometric phases are determined by the metric. We construct a unitary map relating a given time-independent Hilbert space to the time-dependent Hilbert space defined by a positive-definite dynamical invariant. This map defines a transformation that changes the metric of the Hilbert space but leaves the Hamiltonian of the system invariant.

Schrödinger equation is applied in determining the solution of quantum observables which include position of a particle denoted by x, momentum of a particle denoted by p and energy of a particle denoted by ϵ. In this work, properties of operators in Hilbert space have been applied in determining these quantum observables. Note that all quantum observables are self adjoint. An observable may only assume values which belong to the spectrum of its corresponding operator.

4.3.1 Determination of the Probability Position of a Particle using Hilbert Space Operators in three-Dimensition

To determine the position of a particle in three- dimensions using properties of Hilbert space operators, we present the following results.

Theorem 4.4.1

Let $\Psi \in S, c \in \mathbb{C}$. Where S is a vector space. Then

(i) $\Psi^{\star} \in S$

(ii) $(c\Psi)^{\star} = \bar{c}\Psi^{\star}$

(iii) $\Psi^{\star\star} = \Psi$

(iv) If Ψ is invertible then $\Psi^{\star}$ is also invertible and $(\Psi^{\star})^{-1} = (\Psi^{-1})^{\star}$

(v) $\|\Psi\| = \|\Psi^{\star}\|, \|\Psi\Psi^{\star}\| = \|\Psi^{\star}\Psi\| = \|\Psi\|^2 (C^{\star})$

Proof. (i) Sup$\{\|\Psi y\|\}$ =sup $\{| \langle \Psi y, x\rangle |\}$=sup $\{| \langle y, \Psi^{\star} x\rangle |\}$=sup $\{\|\Psi^{\star}\|\}$ So $\Psi^{\star} \in S$ and $\Psi = \Psi^{\star}$. For all $x, y \in \mathcal{H}$

(ii) $\langle (c\Psi)^{\star} y, x\rangle = \langle c\Psi^{\star} y, x\rangle = \langle \Psi^{\star} y, c\bar{x}\rangle = \langle y, \bar{c}\Psi x\rangle$

From the definition of adjoint operator

$$\langle y, c\Psi x\rangle = \langle y, c\Psi^{\star} x\rangle, \text{ thus } (c\Psi) = c\Psi\star, \text{ for all } x, \in \mathcal{H}$$

(iii) $\langle x, \Psi^{\star\star} y\rangle = \langle \Psi^{\star} x, y\rangle = \langle y, \Psi^{\star} x\rangle = \langle x, \Psi\rangle$

Therefore, $\Psi^{\star\star} = \Psi$

(iv) If Ψ is invertible, then $\Psi\Psi^{(-1)} = \Psi^{(-1)}\Psi = I$,

Since $I = I^{\star}$, then $\Psi^{\star}(\Psi^{-1})^{\star} = (\Psi^{-1})^{\star}\Psi^{\star}) = I$, therefore, $\Psi^{\star}$ is invertible and $(\Psi^{-1})^{\star} = (\Psi^{\star})^{-1}$

(v) $\|\Psi\| = \|\Psi^{\star}\|$ follows from (i), $\|\Psi x\|^2 = \langle \Psi x, \Psi x\rangle = \langle \Psi\Psi^{\star} x, x\rangle = \langle x, \Psi^{\star}\Psi x\rangle$

Therefore, $\|\Psi^{\star}\| = \|\Psi^{\star}\Psi\| = \|\Psi\|^2$

□

From the above theorem, the corollary below follows

Corollary 4.4.2

Let $\Psi \in S$, Then:

If $\Psi = \Psi^{\star}$ then Ψ is self adjoint

If $\Psi\Psi^{\star} = \Psi^{\star}\Psi = 1$ then Ψ is unitary

If $\Psi\Psi^\star = \Psi^\star\Psi$ is normal

Recall, the position vector R in three dimension is given by $R = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \sum_j c_j x_j$, where, $c_1 = x, c_2 = y, c_3 = z$, and $x_1 = \mathbf{i}, x_2 = \mathbf{j}, x_3 = \mathbf{k}$, Okelo (2015). In the study of properties of Hilbert space operators and their application in Schrödinger equation, Okelo (2015) used unconventional vector quantity and properties of Hilbert space to express one-dimensional time dependent wave function $\Psi(x,t)$ as a vector $\mid \Psi\rangle$. That is,

$$\langle \Psi \mid \Psi \rangle = \int_{-\infty}^{\infty} \langle \Psi^\star(x,t), \Psi(x,t)\rangle dt = 1 \tag{4.62}$$

If the particle is lying somewhere in a space S, that is, $x \in S$. Then the characteristic function $X_S\ (x) = 1$. If not, then $X_S\ (x) = 0$ and $x \notin S$ (Gagne, 2013). For a particle moving in $\mathbb{R}^3$, the wave function Ψ is in Hilbert space $L^2(R^3)$. If the function $\Psi(x,t)$ of position x at time t, the probability particle to be found at the position x is determined using Gaussian integral as follows:

$$\begin{aligned} pr(\Psi, x) \in S = \textstyle\int_{-\infty}^{\infty} \langle \Psi(x,t) \mid d^3x, \mid, \int_{-\infty}^{\infty} \Psi(x,t)\rangle d^3x \\ = \textstyle\int_{-\infty}^{\infty} \langle \Psi(x,t) \mid . \mid \Psi(x,t)\rangle d^3x \\ = \textstyle\int_{-\infty}^{\infty} \langle \Psi(x,t) \mid . \mid \Psi(x,t)\rangle d^3x \\ = \textstyle\int_{-\infty}^{\infty} \langle \Psi(x,t) \mid \Psi(x,t)\rangle d^3x \\ = \int_{-\infty}^{\infty} \langle \Psi(x,t), \Psi(x,t)\rangle d^3x \in \mathbb{R}^3 \end{aligned} \tag{4.63}$$

Equation (4.62) shows how to determine the probability position of a particle in 3-dimension

4.3.2 Determination of Momentum of a Particle using Operators in Hilbert Space

In quantum mechanics, the momentum $\hat{P}$ is an operator which maps the wave function $\Psi(x,t)$ in the Hilbert space representing quantum state to another function. Recall, momentum operator is defined as,

$$P' = -i\hbar\frac{\partial}{\partial x} \tag{4.64}$$

Using the concept of correlation of properties of Hilbert space operators with momentum, the Fourier transform of $\Psi(x)$ in momentum representation is illustrated as

$$\langle P, \Psi \rangle = \int \langle P, x \rangle \langle x, \Psi \rangle dx. \tag{4.65}$$

Inserting a complete set of momentum states we obtain

$$\langle x, \langle \hat{P}, \hat{\Psi} \rangle \rangle = \int \langle P, x \rangle \langle P, \Psi \rangle \hat{P} dP = \int P \langle x, P \rangle \langle P, \Psi \rangle dp. \tag{4.66}$$

From the equation (4.63) we have

$$\langle P \mid x \mid P \rangle = \frac{\hbar}{i} \frac{\partial}{\partial x} \int \langle x \mid P \rangle. \tag{4.67}$$

Therefore, equation (4.66) becomes

$$\langle x, \langle \hat{P}, \Psi \rangle \rangle = \langle P, x \rangle \langle P, \Psi \rangle dP \int \frac{\hbar}{i} \frac{\partial}{\partial x}. \tag{4.68}$$

It is trivial that $\Psi(x,t) = \Psi_0 e^{-i(kx - \omega t)}$ from equation (4.51). The momentum operator operating on a wave function is given by, $P(\Psi)$. Since both wave function and momentum are vectors, then vector properties hold. These properties include:

(a) The property of scalar product

$$P(\Psi) = \langle P \mid . \mid \Psi \rangle = \Psi P \mid \Psi \rangle = \langle P, \Psi \rangle$$

$$= i\hbar \langle \nabla, \Psi \rangle$$

$$= i\hbar \langle \nabla, \Psi_0 \rangle e^{-i(kx - \omega t)}$$

$$\langle P, \Psi \rangle = \hbar k \Psi$$

$$\langle P, \Psi^\star \Psi \rangle = \hbar k. \tag{4.69}$$

Therefore, $\langle P, \Psi^\star \Psi \rangle = \hbar k$ is a scalar product.

(b) The self adjointness

Since momentum is a quantum observable the property of self adjointness holds. The momentum of a particle is given by the product of mass and velocity. Since velocity is a vector quantity then $v \in S$. Therefore, $p = mv$ and from the above property of scalar product, the momentum operator operating on a wave function can be determined as follows

$$\langle P, \Psi \rangle = i\hbar \langle \nabla, \Psi_0 \rangle e^{-i(kx-\omega t)}. \tag{4.70}$$

4.3.3 Determination of the Energy of a Particle using Operators in Hilbert Space

The operator P is defined as hermitian if its r, s matrix element has the property

$$P_{rs} = \int \Psi_r^\star P \Psi_s d\tau = \int (P\Psi_r)^\star \Psi_r d\tau = \int \Psi_r (P\Psi_r)^\star d\tau = \int [\Psi_s^\star (P\Psi_r^\star)]^\star d\tau = P_{sr}^\star. \tag{4.71}$$

In other words, the matrix elements related by the leading diagonal of P are complex conjugates of each other. Operators that are Hermitian enjoy certain properties. The Hamiltonian (energy) operator is Hermitian, and so are the various angular momentum operators. In order to show this, first recall that the Hamiltonian is composed of a kinetic energy part which is essentially $\frac{P^2}{2m}$ and a set of potential energy terms which involve the distance coordinates x, y etc. If we can prove that the various terms comprising the Hamiltonian are Hermitian then the whole Hamiltonian is Hermitian.

Energy is quantum observable, therefore the property of self adjointness holds. The energy of a particle is an eigenvalue of Hamiltonian operator. In quantum mechanics, the Hamiltonian operator is the sum of kinetic energy and potential energy.

$$\hat{H} = \hat{K.E} + \hat{P.E} \tag{4.72}$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{P.E}, \tag{4.73}$$

where $\hat{p} = -i\hbar \frac{\partial}{\partial x}$
Therefore, $K.E = -\frac{\hbar^2}{2m} \langle \nabla_x \nabla_x, \Psi(x,t) \rangle$ for a particle in 1dimension (Wei, 2016) . Generally, the potential energy is denoted by $\hat{V}$, therefore the total energy of a particle in three dimension is be given by

$$\langle H, \Psi \rangle = -\frac{\hbar^2}{2m} \langle \langle \nabla, \nabla \rangle, \Psi(x,t) \rangle + \langle V, \Psi(x,t) \rangle. \tag{4.74}$$

CHAPTER FIVE
CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

Operators in Hilbert space have been studied in the past decades. These operators possess properties which are of interest study in Quantum Mechanics. Schrödinger equation is key in the study of Quantum Mechanics. This study focused on derivation of Schrödinger equation using operators Hilbert space and we applied properties of Reimann's integral, Neumann's integral, Lebesque Integral and the properties of Hilbert space operators. From these abstract Mathematical properties, we obtained the Schrödinger equation as

$$\tfrac{-\hbar^2}{2m}\langle \nabla, \nabla \rangle, \Phi\rangle = i\hbar\tfrac{\partial}{\partial t}\Phi. \text{ (Refer to equation (4.61)}$$

The correlation between operators in Hilbert space and electromagnetic wave equation is established by utilizing the Maxwell's equations as well as the properties of operators in Hilbert space which include:

(i) Orthogonally property $\langle x, y\rangle = 0$

(ii) Self- adjointness $\langle x, y\rangle = \langle \overline{y, x}\rangle$

(iii) Unitary $\langle x, x^{\star}\rangle = 1$ iff $x = x^{\star}$

(iv) Positivity $\langle x, x\rangle \geq 0 = \|x\|^2$

(v) Nullity property $\langle x, x\rangle = 0$ iff $x = 0$

Subsequently, using the electromagnetic wave equation and properties of Hilbert space operators Schrödinger equation was derived as illustrated in section (4.3).

Similarly, the correlation between Einstein theory of relativity and Hilbert space operators was established and the results obtained were used in the derivation of Schrödinger equation as an alternative approach.

On the application of operators in Hilbert space on Quantum observables i.e. energy, momentum and position, properties of Hilbert space operators are applied. These observables are Hermitians operator which are one of the operators of Hilbert spaces. Other operators in

Hilbert space have interesting features which make them applicable in describing quantum observables. Hilbert space considers a function as point belonging to an infinite space.

In determining the probability position of a particle, the existing literature on the probability position of a particle in a space is used as well as in a confined in a potential barrier is highly considered. From the result the position of a particle in three- dimensions is applying the properties of Hilbert space operators can be determined using the formula

$$\int_{-\infty}^{\infty}\langle\Psi(x,t),\Psi(x,t)\rangle d^3x \in R^3 \text{ (refer to equation (4.62))}$$

Total energy which is the second quantum observable is taken as a Hermitian operator which is Hamiltonian. From our result, it is represented by the equation.

$$\langle H,\Psi\rangle = -\frac{\hbar^2}{2m}\langle U\nabla,\nabla\rangle,\Psi(x,t)\rangle + \langle V,\Psi(x,t)\rangle \text{ (refer to equation (4.74))}$$

The other quantum observable is momentum which from the literature is defined as the product of mass and velocity. The momentum operator is represented by $\hat{p}$ which is also Hermitian.

5.2 Recommendations for Further Research

In this study, we have determined alternative derivation of Schrödinger equation using properties of Hilbert space operators. Further the study has described the quantum observables of a particle as solutions to the Schrodinger equation in Hilbert spaces. The study recommends the derivation of of Schrodinger equation of many body intersecting particles and description of quantum observables in Hilbert spaces.

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Appendix A
Research Clearance and Authorization(Chuka University)

CHUKA UNIVERSITY

Knowledge is Wealth *(Sapientia divitia est)* Akili ni Mali

CHUKA UNIVERSITY INSTITUTIONAL ETHICS REVIEW COMMITTEE

Telephones: 020-2310512/18 P. O. Box 109-60400, Chuka
Direct Line: 0772894438 Email: info@chuka.ac.ke, Website: www.chuka.ac.ke

REF: CUIERC/ NACOSTI/010 **8TH JULY 2019**
TO: ELIZABETH MBATHA

Dear Sir/madam

RE: FORMULATION OF SCHRODINGER EQUATION USING THE HILBERT SPACE OPERATOR APPROACH.

This is to inform you that ***Chuka University IERC*** has reviewed and approved your above research proposal. Your application approval number is ***NACOSTI/NBC/AC-0812***. The approval period is ***1st April, 2019 – 31st March, 2020.***

This approval is subject to compliance with the following requirements;

viii. Only approved documents including (informed consents, study instruments, MTA) will be used
ix. All changes including (amendments, deviations, and violations) are submitted for review and approval by ***Chuka University IERC***.
x. Death and life threatening problems and serious adverse events or unexpected adverse events whether related or unrelated to the study must be reported to ***Chuka University IERC*** within 72 hours of notification
xi. Any changes, anticipated or otherwise that may increase the risks or affected safety or welfare of study participants and others or affect the integrity of the research must be reported to ***Chuka University IERC*** within 72 hours
xii. Clearance for export of biological specimens must be obtained from relevant institutions.
xiii. Submission of a request for renewal of approval at least 60 days prior to expiry of the approval period. Attach a comprehensive progress report to support the renewal.
xiv. Submission of an executive summary report within 90 days upon completion of the study to ***Chuka University IERC***.

Prior to commencing your study, you will be expected to obtain a research license from National Commission for Science, Technology and Innovation (NACOSTI) https://oris.nacosti.go.ke and also obtain other clearances needed.

Yours sincerely

PROF. ADIEL MAGANA
CHAIRMAN CHUKA UNIVERSITY IERC

Chuka University is ISO 9001:2015 Certified... *Inspiring Environmental Sustainability for Better Life*

Appendix B
Research Authorization(NACOSTI)

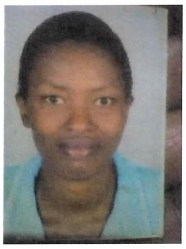
REPUBLIC OF KENYA

NATIONAL COMMISSION FOR SCIENCE,TECHNOLOGY & INNOVATION

Ref No: **973985**

Date of Issue: **30/August/2019**

RESEARCH LICENSE



This is to Certify that Miss.. ELIZABETH MBATHA of CHUKA UNIVERSITY, has been licensed to conduct research in Tharaka-Nithi on the topic: formulation of Schrodinger Equation using Hilbert Space Operators for the period ending : 30/August/2020.

License No: **NACOSTI/P/19/79**

973985

Applicant Identification Number

Director General
NATIONAL COMMISSION FOR SCIENCE,TECHNOLOGY & INNOVATION

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