CHUKA UNIVERSITY

UNIVERSITY EXAMINATIONS

EXAMINATION FOR THE AWARD OF DEGREE OF MASTER OF SCIENCE IN PURE MATHEMATICS

MATH 805: ABSTRACT INTEGRATION 1

STREAMS: MSC (MATHS) **TIME: 3 HOURS**

DAY/DATE: FRIDAY 06/12/2019 **11.30 A.M. – 2.30 P.M.**

INSTRUCTIONS:

- **Answer ANY THREE questions.**
- **Do not write on the question paper.**

QUESTION ONE: (20 MARKS)

(a) Let $E_n \subseteq \mathbb{R}\ \forall n \in \mathbf{N}$, prove that the set function $M^*: \wp(\mathbb{R}) \to \mathbb{R}^* \geq 0$ is finitely subadditive. i.e.
$\mathcal{M}^*\left(\bigcup_{n=1}^{k} E_n\right) \leq \sum_n^k \mathcal{M}^*(E_n)$ (3 marks)

(b) Let E be an atmost countable set (i.e. suppose E is the set of all rational numbers, $\mathbf{Q}$), prove that $\mathcal{M}^*[\mathbf{Q}] = 0$ (2 marks)

(c) Define a non-degenerate interval I. Hence show that every non-degenerate interval is countable (3 marks)

(d) Let ρ be the set of all intervals of $\mathbb{R}$ and $\wp(\mathbb{R})$ the class of all subsets of $\mathbb{R}$. Suppose the set function $\lambda: \sigma \to \mathbb{R}^* \geq 0$ represent a length function for a non-negative real number $\lambda(I)$, and $\mathcal{M}^*: \wp(\mathbb{R}) \to \mathbb{R}^* \geq 0$ be the Outer Lebesgues measure of a subset E of $\mathbb{R}$. Prove that M^* is an extension of λ. i.e.
$M^*[I] = \lambda(I)\ \forall\ I \in \rho$ (12 marks)

QUESTION TWO: (20 MARKS)

(a) Prove that if E is non-Lebesque measurable subset of $\mathbb{R}$, then there exists a subset A of E such that $0 < \mathcal{M}^*[A] < \infty$ (5 marks)

(b) By constructing the Cantor's set p, prove that this set is Lebesque measurable (9 marks)

(c) Let X, Y be non-void sets and $f: \mathrm{X} \to Y$ be a function. Let $\beth$ be the σ- algebra of subsets of Y and let $\mathfrak{x} = \{f^{-1}(E): E \in \beth\}$. Prove that then $\mathfrak{x}$ is the σ- algebra of subsets of X (6 marks)

QUESTION THREE: (20 MARKS)

(a) Distinguish an almost everywhere convergence and an almost uniform convergence. Hence state and prove the Egoroff's Theorem (6 marks)

(b) State and prove Fatou's Lemma (7 marks)

(c) (i) Prove that the cardinality of the Borel set, Card $\boldsymbol{\mathcal{B}}(\mathbb{R}) = \mathrm{c}$ (4 marks)

(ii) Hence show that the Borel set $\boldsymbol{\mathcal{B}}(\mathbb{R})$ is a proper subset of a Lebesque measurable set (3 marks)

QUESTION FOUR: (20 MARKS)

(a) Let $(X, \mathfrak{x}, \mu)$ be a complete measure space and f, g be functions defined $\mu.a.e$ on X, such that $f \equiv g\ \mu.a.e$. Prove that if f is $\mathfrak{x}$-measurable, so is g. (6 marks)

(b) State without proof the Approximation Theorem in measurable spaces (2 marks)

(c) Let $f: \mathrm{X} \to \mathbb{R}^*$ be a function, define f^+, f^- as positive and negative parts of f respectively, show that $f = f^+ - f^-$

(4 marks)

(d) Let $(X, \mathfrak{x})$ be a measurable space and $f: \mathrm{X} \to \mathbb{C}$ a function with $f = f_1 + if_2$, where $f_1 = \mathbb{R}ef, f_2 = \mathrm{im}f$ and $i = \sqrt{-1}$. Show that the following statements are equivalent:

(i) f is $\mathfrak{x}$-measurable

(ii) f_1, f_2 are both is $\mathfrak{x}$-measurable (8 marks)
