CHUKA UNIVERSITY

UNIVERSITY EXAMINATIONS

FIRST YEAR EXAMINATION FOR THE AWARD OF DEGREE OF MASTER OF SCIENCE IN PURE MATHEMATICS

MATH 801: FUNCTIONAL ANALYSIS I

STREAMS: **TIME: 3 HOURS**

DAY/DATE: WEDNESDAY 4/12/2019 **11.30 A.M – 2.30 P.M**

INSTRUCTIONS

Answer ANY THREE Questions.
Do not write on the question paper.

QUESTION ONE: (20 MARKS)

(a) (i) State and prove Holder's and Minkowski's inequalities as applied in metric spaces. (6marks)

(ii) Hence show that $\forall\, x, y \in \mathbb{R}^n$, a function $d: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$d(x,y) = \left(\left(\sum_{i=1}^{n}(x_i - y_i)^2\right)\right)^{\frac{1}{2}}$ is a complete metric space. (5marks)

(iii)Show that an open ball is an open set in a metric space (3marks)

(b) Let X be a space of all bounded continous real valued functions defined on the interval $(0,1)$ and let $d(x,y) = \int_0^1 |\, x(t) - y(t)|dt\ \ \forall\, x, y \in X$. Show that (X, d) is not complete. (3marks)

(c) Let (X, d) be a metric space and $x, y, z, u \in (X, d)$. Show that

$$|\, d(x,y) - d(z,u)| \leq d(x,z) + d(y,u)$$ (3marks)

QUESTION TWO: (20 MARKS)

(a) Let M be a non-empty subset of a metric space (X, d) and $\bar{M}$ be the closure of M. Prove that

$x \in \bar{M}$ if and only if there exists sequence $(x_n) \in M: x_n \to x$ (3marks)

(b) Prove that a subspace M of a complete metric space (X, d) is itself complete iff the set M is closed in X (3marks)

(c) State and prove Baires Category theorem for metric spaces (5marks)

(d) Prove that any contraction mapping f of a non-empty complete metric space (X, d) into itself has a unique fixed point. (5marks)

(e) Prove that every compact subset F of a metric space(X, d) is closed (4marks)

QUESTION THREE: (20 MARKS)

(a) Let X, Y be vector spaces and $T: X \to Y$with $D(T) \subset X \ \ and \ R(T) \subset Y$. Show that

(i) $T^{-1}: R(T) \to D(T)$exists if and only if $Tx = 0 \Longrightarrow x = 0$

(ii) If T^{-1} exists then it is linear (5marks)

(b) Prove that a real matrix $A = a_{jk}$ with r rows and n columns defines a bounded operator $T: \mathbb{R}^n \to \mathbb{R}^r$ by means of $y = A_x$ (4marks)

(c) Let $T: D(T) \subset X \to Y$ be a linear operator and X, Y are normed spaces. Prove that T is continous if and only if it is bounded (3marks)

(d) State without prove the Generalized(Complex version) of the Hahn Banach Extension Theorem. Hence discuss the application of this theorem to the following contexts

(i) To the second dual

(ii) In generalization of the classical Theorem of Liouville (8marks)

QUESTION FOUR: (20 MARKS)

(a) Define a seminorm P on a vector space X and hence show that a seminorm is a convex set on the vector space X (4marks)

(b) State and prove the Banach Alaoglu Theorem. (5marks)

(c) Let M be a closed subspace of a normed space X. Define the quotient space $X\backslash_M$ with the norm by $||x + M|| = inf \{||x + m\}: m \in M$, which is also a normed space. Prove that if X is a Banach space, then $X\backslash_M$ is also a Banach space. (5marks)

(d) Distinguish an absolutely convex set and an absorbent set on a vector space X. Hence prove that if P is a seminorm on a vector space X and for $r > 0$, the sets $S = \{x \in X: P(x) \leq r\}$ and $T = \{x \in X: P(x) < r\}$ are absolutely convex and absorbent (6marks)
