CHUKA UNIVERSITY

UNIVERSITY EXAMINATIONS

EXAMINATION FOR THE AWARD OF DEGREE OF MASTER OF SCIENCE IN PURE MATHEMATICS

MATH 802: FUNCTIONAL ANALYSIS II

STREAMS: MSC (PURE MATH) P/T **TIME: 3 HOURS**

DAY/DATE: TUESDAY 14/04/2020 **2.30 PM – 5.30 PM**

INSTRUCTIONS:

- Answer any three questions
- Sketch maps and diagrams may be used whenever they help to illustrate your answer
- Do not write on the question paper
- This is a **closed book exam,** No reference materials are allowed in the examination room
- There will be **No** use of mobile phones or any other unauthorized materials
- Write your answers legibly and use your time wisely

QUESTION ONE (20 MARKS)

(a) Define what is meant by a compact operator T on a Hilbert space H. Hence show that every compact operator T on H is bounded. [4 marks]

(b) State the parallelogram law on an inner product space H. [2 marks]

(c) Show that every unitary operator is normal but the converse is not necessarily true. [4 marks]

(d) Let H be a Hilbert space and y be a fixed vector of H. Prove that the functional $f(x) =< x, y > \forall\, x \,\epsilon\, H$ is a continuous linear functional. [3 marks]

(e) Suppose that $x_n \rightharpoonup x \; and \; \|x_n\| \to \|x\|$. Show that x_n converges to x strongly. [3 marks]

(f) Define what is meant by each of the following

(i) Reflexive Space X [2 marks]

(ii) Uniformly convex Banach space X [2 marks]

QUESTION TWO (20 MARKS)

(a) State and prove Riesz Representation theorem on a Hilbert space H. [10 marks]

(b) Let $T: X \to Y$ be bounded linear operator on a Hilbert space X into itself. Prove each of the following:

(i) $\|T\| = \|T^*\|$ [3 marks]

(ii) $\|T^*T\| = \|T\|^2$ [3 marks]

(c) Let $T: X \to X$ be bounded linear operator on a Hilbert space X. Prove that T is self-adjoint if and only if $< Tx, x >$ is real. [4 marks]

QUESTION THREE (20 MARKS)

(a) Let M and N be closed subspaces of a Hilbert space H such that $M \perp N$. Show that the subspaces given by: $M + N = \{x + y \epsilon H : X \epsilon M, \ y \in N\}$ is also closed. [8 marks]

(b) Let y and z be fixed elements of a Hilbert Space H and define $T: H \to H \ by \ Tx =< Tx, x > z$. Show that T is compact. [5 marks]

(c) Prove that the sequence $\{x_n\}$ in a Hilbert space H is weakly convergent to the limit $x \epsilon H \ iff \lim_{n\to\infty} < x_n, z >=< x, z > \forall z \in H$. [4 marks]

(d) Let l_p^n be a normed space of all n-tuples $x = (x_1, x_2, \dots x_n)$ of real numbers equipped with the norm, $\|x\| = (\sum_k^n = 1|x_k|^p)^{\frac{1}{p}}, 1 \leq p \leq \infty$. Show that l_p^n is not an inner product space of $p \neq 2$. [3 marks]

QUESTION FOUR (20 MARKS)

(a) State without proof Uniformly Bounded Principle. [2 marks]

(b) Let X and Y be Banach spaces and $T \in B(X, Y)$. Suppose that T is surjective then show that T is an open napping. [4 marks]

(c) A linear operator T is closed if and only if its graph of T, G_T, is a closed subspace. [6 marks]

(d) Prove that the canonical mapping J given by $J: X \to X^{**}, x \to \emptyset x$ is an isomorphism from normed space X onto the normed space X onto the range of J, R(J). [4 marks]

(e) Prove that every inner product space H is uniformly complex. [4 marks]
