CHUKA **UNIVERSITY**

UNIVERSITY EXAMINATIONS

FIRST YEAR EXAMINATION FOR THE AWARD DEGREE OF SCIENCE IN MATHEMATICS (PURE)

MATH 806: ABSTRACT INTEGRATION II

STREAM: MATH Y1S2 **TIME: 3 HOURS**

DAY/DATE: THURSDAY 9/04/2020 **11.30 A.M - 230 P.M.**

INSTRUCTIONS:

Answer ANY THREE Questions.

QUESTION ONE: (20 MARKS)

(a) State without proof

(i) The Radon-Nikodym Theorem (2 Marks)

(ii) Fubini's Theorem (2 Marks)

(b) Let $(X, \mathfrak{x}, \mu)$ be a measure space and f be a measurable function on X for which $\int f d\mu$.

Prove that the set function $\nu: \mathfrak{x} \to \mathbb{R}^*$ defined by $\nu(E) = \int_E f d\mu, \forall E \in \mathfrak{x}$ is a signed measure. (4 Marks)

(c) Let ν a signed measure on $(X, \mathfrak{x})$ and $E \in \mathfrak{x}$ such that $0 < \nu(\mathrm{E}) < \infty$. Prove that there exists a measurable subset A of E which is positive with respect to ν and more so $0 < \nu(\mathrm{E}) < \infty$ (12 Marks)

QUESTION TWO: (20 MARKS)

(a) Let $(X, \mathfrak{x})$ be a measurable space and ν, μ be finite measures on $\mathfrak{x}$. Prove that $\nu << \mu$, that is, ν is absolutely continuous with respect to μ if and only if for each $\varepsilon > 0$ $\ni \delta > 0$ such that for $E \in \mathfrak{x}$ and $\mu(E) < \varepsilon$ it implies $\nu(E) < \delta$ (6 Marks)

(b) Prove that if the set E is positive with respect to the measure ν it implies that every measurable subset A of E has a positive measure, $\nu(\mathrm{A}) \geq 0$ but the converse is not necessarily true. (4 Marks)

(c) Let $(X, \mathfrak{x})$ be a measurable space and ν a signed measure on $\mathfrak{x}$. Define the Jordan Decomposition of ν. Hence show that the Jordan Decomposition of ν is unique, that is, each signed measure ν on $(X, \mathfrak{x})$ has a unique Jordan representation, namely $\{\nu^+, \nu^-\}$ (10 Marks)

QUESTION THREE: (20 MARKS)

(a) State and prove the Lebesque's Decomposition Theorem (14 Marks)

(b) Illustrate using an appropriate example that the measure μ is finite is a necessary condition in the Radon-Nikodym Theorem. (6 Marks)

QUESTION FOUR: (20 MARKS)

(a) Let X be a non-void set and $\mathfrak{A}$ an algebra of subsets of X and μ a measure on the algebra. Let μ^* be the outer measure induced by μ. Show that μ^*extends μ on. Hence state without proof the Caratheodory's Extension Theorem (10 Marks)

(b) Let X, Y be non-void sets. Define the following

(i) A measurable rectangle of $X \times Y$

(ii) An elementary set E in $X \times Y$

(iii) The X –section of $E \subseteq X \times Y$ (3 Marks)

(c) Define a monotone class on a non-void set X. Prove that the σ-algebra generated by the class R of all measurable rectangles of $X \times Y$ is the smallest monotonic class containing the set of all elementary subsets of $X \times Y$ (7 Marks)

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